Calculus I: MAC2311
Spring 2024
Exam 3 A
4/4/2024
Time Limit: 100 Minutes

Name: Key

Section: ____________

UF-ID: ____________

Scantron Instruction: This exam uses a scantron. Follow the instructions listed on this page to fill out the scantron.

A. Sign your scantron **on the back** at the bottom in the white area.

B. Write **and code** in the spaces indicated:

1) Name (last name, first initial, middle initial)
2) UFID Number
3) 4-digit Section Number

C. Under *special codes*, code in the test numbers 3, 1:

1 2 • 4 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, fill in the *Test Form Code* as A .

• B C D E

E. This exam consists of 14 multiple choice questions and 4 free response questions. Make sure you check for errors in the number of questions your exam contains.

F. The time allowed is 100 minutes.

G. WHEN YOU ARE FINISHED:

1) Before turning in your test check for **transcribing errors**. Any mistakes you leave in are there to stay!

2) You must turn in your scantron and free response packet to your proctor. **Be prepared to show your proctor a valid GatorOne ID or other signed ID.**

It is your responsibility to ensure that your test has **18 questions**. If it does not, show it to your proctor immediately. You will not be permitted to make up any problems omitted from your test after the testing period ends. There are a total of 105 points available on this exam.

Part I Instructions: 14 multiple choice questions. Complete the scantron sheet provided with your information and fill in the appropriate spaces to answer your questions. Only the answer on the scantron sheet will be graded. Each problem is worth five (5) points for a total of 70 points on Part I.

$f(x)$

1. Let $y = 2x^2 + x$. Calculate $dy + \Delta y$ as x goes from 2 to 1.

x_1 x_2

(A) -12 (B) 12 (C) -16

(D) 2 (E) -9

$dx = x_2 - x_1 = 1 - 2 = -1$

$dy = f'(x)\,dx$

$dy = (4x+1)dx = (4(2)+1)(-1) = -9$

$\Delta y = y_2 - y_1 = (2(1)^2+1) - (2(2)^2+2) = 3 - 10 = -7$

$dy + \Delta y = -9 + (-7) = \boxed{-16}$

2. Find the linearization, $L(x)$, of $f(x) = \sqrt[3]{x+5}$ at $a = 3$.

$f(x) = (x+5)^{1/3}$

(A) $2 + \frac{1}{4}(x-3)$ (B) $2 + \frac{1}{12}(x-3)$ (C) $2 - \frac{1}{12}(x-3)$

(D) $2 + \frac{1}{4}(x+3)$ (E) $2 + \frac{1}{6}(x+3)$

$f'(x) = \dfrac{1}{3(x+5)^{2/3}}$, $f'(3) = \dfrac{1}{3(8)^{2/3}} = \dfrac{1}{3\cdot 4} = \dfrac{1}{12}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(3) + f'(3)(x-3)$

$L(x) = 2 + \frac{1}{12}(x-3)$

3. Find the critical numbers of $f(x) = x^{\frac{2}{3}}(2x - 1)$. Domain: $(-\infty, \infty)$

(A) $x = 0$ (B) $x = \frac{1}{5}$ (C) $x = 0, \frac{1}{5}$ (D) $x = 0, \frac{1}{2}$ (E) $x = \frac{1}{2}$

$$f'(x) = x^{2/3} \cdot (2) + \frac{2}{3}x^{-1/3}(2x-1)$$

$$f'(x) = x^{-1/3}\left(2x + \frac{2}{3}(2x-1)\right)$$

$$f'(x) = \frac{2x + \frac{4x}{3} - \frac{2}{3}}{x^{1/3}}$$

$$f'(x) = \frac{\frac{10}{3}x - \frac{2}{3}}{x^{1/3}}$$

$$\frac{10x}{3} - \frac{2}{3} = 0$$

$$10x - 2 = 0$$

$$x = \frac{1}{5}$$

$$x^{1/3} = 0$$

$$x = 0$$

4. Let $g(x) = x^3 - 12x + 23$. Calculate the absolute maximum of $g(x)$ on the interval $[-5, 3]$.

a b

(A) 20 (B) 18 (C) 30 (D) 39 (E) 44

$$g'(x) = 3x^2 - 12$$

$$3x^2 - 12 = 0$$

$$3(x^2 - 4) = 0$$

$$3(x-2)(x+2) = 0$$

$$x = 2, \quad x = -2$$

Test endpoints and critical #'s that belong to the interval $[-5, 3]$.

$f(-5) = -42$

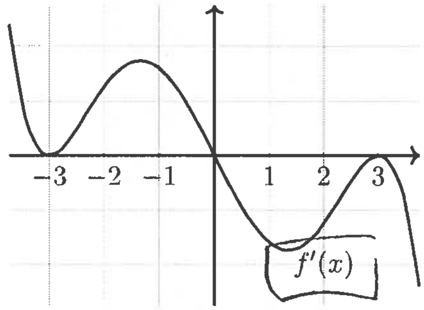
$f(-2) = 39$ largest

$f(2) = 7$

$f(3) = 14$

5. The graph of the **first derivative** $f'(x)$ of a function $f(x)$ is shown below. On which interval(s) is $f(x)$ increasing?

increasing → $f'(x) > 0$
we need the intervals where this graph is positive.



(A) $(-\infty, -3)$ (B) $(-\infty, -3) \cup (-3, 0)$ (C) $(-3, -1)$ (D) $(-3, 0) \cup (1, 3)$ (E) $(0, 3)$

6. Let $f(x) = x^2 - x^{\frac{2}{3}}$. Are the assumptions for the Mean Value Theorem met for $f(x)$ on the interval $[-1, 8]$?

(A) No, since $f(x)$ is not continuous on $[-1, 8]$.

(B) No, since $f(x)$ is not differentiable on $(-1, 8)$.

(C) No, since $f(-1) \neq f(8)$.

(D) Yes, $f(x)$ meets all assumptions required for the Mean Value Theorem.

$f'(x) = 2x - \dfrac{2}{3x^{1/3}}$

$f'(0)$ is undefined and 0 is in the interval $(-1, 8)$.

Not differentiable @ $x = 0$.

7. Find the number c which satisfies Rolle's Theorem for $f(x) = \sin x$ on $[0, \pi]$.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$

(D) $\frac{3\pi}{2}$ (E) π

$f'(x) = \cos x$

$0 = \cos x$

$x = \frac{\pi}{2}$

8. Find the interval(s) where $f(x) = \sqrt{\frac{x}{x-5}}$ is **decreasing**. Domain: $(-\infty, 0] \cup (5, \infty)$

(A) $(-\infty, 0) \cup (5, \infty)$ (B) $(0, 5)$ (C) $(-\infty, \infty)$

(D) $(-\infty, 0)$ (E) $(0, 5) \cup (5, \infty)$

$f(x) = \left(\frac{x}{x-5}\right)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left(\frac{x}{x-5}\right)^{-\frac{1}{2}} \left[\frac{x-5-x}{(x-5)^2}\right]$

$f'(x) = \frac{1}{2}\left(\frac{x-5}{x}\right)^{\frac{1}{2}} \left[\frac{-5}{(x-5)^2}\right]$

always positive always negative

$\therefore$ $f'(x)$ is always negative.

9. Let x and y be two positive numbers whose product is 100. What is the minimum sum for two such numbers?

$x>0, y>0$

(*A*) 10 (*B*) 20 (*C*) 25 (*D*) 40 (*E*) None of the above.

$xy = 100 \rightarrow x = \frac{100}{y}$

$x = \frac{100}{10} = 10$

minimize: $F = x + y$

$x + y = 10 + 10 = \boxed{20}$

$F = \frac{100}{y} + y$

$F' = -\frac{100}{y^2} + 1 = 0$

$1 = \frac{100}{y^2}$

$y = \pm 10$, but $y > 0$. So $\boxed{y = 10}$

10. Which of the following correctly describes $f(x) = 2 + 2x^2 - x^4$?

(*A*) $f(x)$ has two local maxima and one local minimum.

(*B*) $f(x)$ has one local maximum and one local minimum.

(*C*) $f(x)$ has two local maxima and two local minima.

(*D*) $f(x)$ has one local maximum and two local minima.

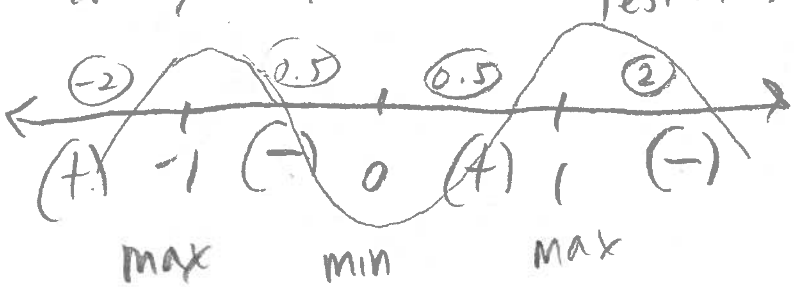
$f'(x) = 4x - 4x^3$

$f'(x) = 4x(1-x^2)$

$f'(x) = 4x(1-x)(1+x)$

$4x(1-x)(1+x) = 0$

$x=0,\ x=1,\ x=-1$ Test in $f'(x)$:



11. Let $f(x) = \sin^2 x$. Determine the interval(s) on $[0, \pi]$ where $f(x)$ is concave down. need $f''(x)$

(A) $[0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$ (B) $(\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{3\pi}{4}, \pi]$ (C) $(\frac{\pi}{4}, \frac{3\pi}{4})$

(D) $[0, \frac{\pi}{2})$ (E) $(\frac{\pi}{2}, \pi]$

$f'(x) = 2\sin x \cos x$

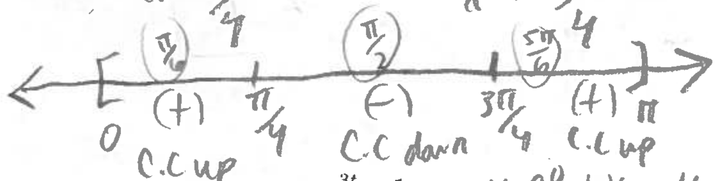
$f''(x) = 2[\sin x(-\sin x) + (\cos x)(\cos x)]$

$f''(x) = 2[\cos^2 x - \sin^2 x]$ ← difference of squares

$2(\cos x - \sin x)(\cos x + \sin x) = 0$ ← Find solutions in $[0, \pi]$

$\cos x = \sin x$ $\cos x = -\sin x$ Test in $f''(x)$.

$x = \pi/4$ $x = 3\pi/4$



12. Evaluate $\lim_{t \to 0} \frac{e^{3t} - 1}{t}$ $=$ "$\frac{e^0 - 1}{0}$" $=$ "$\frac{0}{0}$" ✓

(A) 0 (B) 1 (C) 3 (D) ∞ (E) Does not exist.

$\overset{\mathcal{L}}{=} \lim_{t \to 0} \frac{3e^{3t}}{1} = 3e^0 = \boxed{3}$

13. Evaluate $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$ "$\infty \cdot \tan(\frac{1}{\infty})$" = "$\infty \cdot 0$"

(A) 0 (B) 1 (C) −1

(D) ∞ (E) $-\infty$

$= \lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}}$ "$\frac{0}{0}$" ✓

$\overset{L}{=} \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x})\cdot\cancel{(\frac{-1}{x^2})}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty} \sec^2(\frac{1}{x})$

$= \sec^2(0)$

$= (\sec(0))^2 = 1^2 = \boxed{1}$

14. Which of the following functions is almost **always** concave down?

(A) $f(x) = -3x^3$ (B) $f(x) = x^4$ (C) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ (D) $f(x) = \sin x$ (E) $f(x) = -4x^4$

We need an answer choice where $f''(x)$ is always negative.

Look at (E):

$f'(x) = -16x^3$

$f''(x) = -48x^2 \leq 0$

Given any nonzero value of x, f(x) will be concave down.

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Part II Instructions: 4 free response questions. Neatly give a complete solution to each problem and show all work and intermediate steps. We are grading the work and notation as well as the answer. Problems 1 through 3 are worth seven (7) points each. Problem 4 is worth fourteen (14) points. A total of 35 points is possible on Part II. **No credit will given without proper work.** If we cannot read it and follow it, you will receive no credit for the problem.

For Instructor Use Only:

FR 1	
FR 2	
FR 3	
FR 4	
Total Points	

1. Clearly state the indeterminate form of the limit below. After this, evaluate the limit, stating the use of L' Hopital's rule wherever applicable.

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} \to \text{"}(1+0)^{\infty}\text{"}$$

$$\text{"}1^{\infty}\text{"}$$

$$e^{\lim_{x\to 0^+} \ln(1+\sin 4x)^{\cot x}}$$

$$= e^{\lim_{x\to 0^+} \cot(x)\cdot\ln(1+\sin 4x)}$$

Let's look at:

$$\lim_{x\to 0^+} \cot(x)\cdot\ln(1+\sin(4x))$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\tan(x)} \quad \text{"}\frac{0}{0}\text{"} \checkmark$$

$$\overset{\mathcal{L}}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(4x)}\cdot 4\cos(4x)}{\sec^2(x)} = \frac{\frac{4\cos(0)}{1+\sin(0)}}{\sec^2(0)} = \frac{\frac{4}{1}}{1} = 4$$

Therefore,

$$e^{\lim_{x\to 0^+} \cot(x)\cdot\ln(1+\sin(4x))} = \boxed{e^4}$$

See form B for an alternate solution.

2. Find the absolute maximum and minimum of $f(x) = x\sqrt{4-x^2}$ on the interval $[-1, 2]$.

$f(x) = x(4-x^2)^{1/2}$

$f'(x) = (4-x^2)^{1/2} + \frac{1}{2}(4-x^2)^{-1/2}(-2x)(x)$

$f'(x) = (4-x^2)^{1/2} - x^2(4-x^2)^{-1/2}$

$f'(x) = (4-x^2)^{-1/2}\left[4-x^2-x^2\right]$

$f'(x) = \dfrac{4-2x^2}{\sqrt{4-x^2}}$

Find critical points in interval

$f'(x) = 0$

$\therefore\ 4-2x^2=0$

$4 = 2x^2$

$2 = x^2$

$\pm\sqrt{2} = x$

$x = -\sqrt{2}$ not in interval

$f'(x)$ undefined

$\therefore\ 4-x^2=0$

$4 = x^2$

$x = \pm 2$

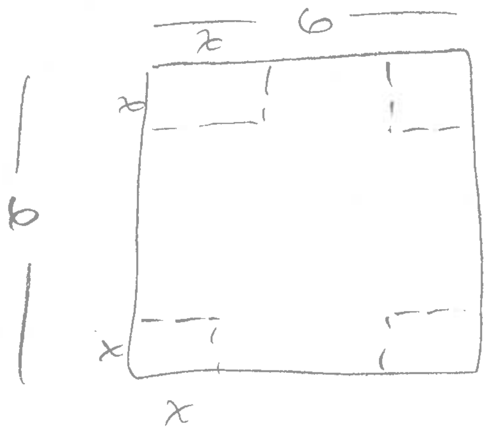
$x = -2$ not in interval

$f(-1) = -1\sqrt{3} = -\sqrt{3}$ abs min

$f(\sqrt{2}) = \sqrt{2}\sqrt{2} = 2$ abs max

$f(2) = 2\sqrt{0} = 0$

3. A box with an open top is to be constructed from a square piece of cardboard that is six feet on each side, by cutting out a square from each of the four corners and bending up the sides. What is the largest volume that such a box can have? Show all work. You may wish to draw a diagram to support your work. Recall $V = lwh$.



$V = l \cdot w \cdot h$

$V = (6-2x)(6-2x)(x)$

$V = (36 - 12x - 12x + 4x^2)\,x$

$V = 36x - 24x^2 + 4x^3$

$V' = 36 - 48x + 12x^2$

Need $V' = 0$

$12x^2 - 48x + 36 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$\boxed{x=1}$, $\cancel{x=3}$

$x=3$ is too big of a cut

Largest Volume

$V = (6-2(1))(6-2(1))(1) = 4^2 = \boxed{16 \text{ un}^3}$

4. Let $f(x) = \frac{x-1}{x^2}$.

Part 1. List the function's domain, intercepts, and vertical or horizontal asymptotes.

Domain: $(-\infty, 0) \cup (0, \infty)$

intercepts: $(1, 0)$

Vertical asymptote $\boxed{x = 0}$

Horizontal asymptote: $\lim\limits_{x \to \infty} \frac{x-1}{x^2} = 0$, $\lim\limits_{x \to -\infty} \frac{x-1}{x^2} = 0$ $(y = 0)$

Part 2. Use the first derivative to find the intervals where the function is increasing/decreasing. Also give the x values at which any local maxima or minima occur and their corresponding y-values.

$$f'(x) = \frac{2-x}{x^3}$$

$f'(x) = 0$

$2 - x = 0$

$x = 2$

$f'(x)$ undefined

$x = 0$

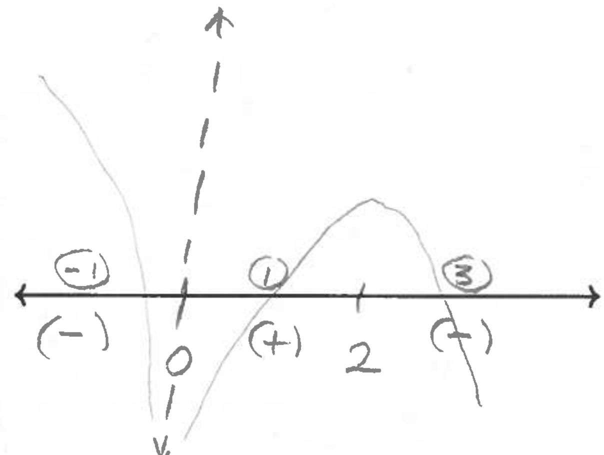
increasing: $(0, 2)$

decreasing: $(-\infty, 0) \cup (2, \infty)$

local max: $(2, \frac{1}{4})$

no local min

f'



Part 3. Find the intervals where the function is concave up/concave down and points of inflection.

$$f''(x) = \frac{2(x-3)}{x^4}$$

always positive

$f''(x) = 0$

$x - 3 = 0$

$x = 3$

$f''(x)$ undefined

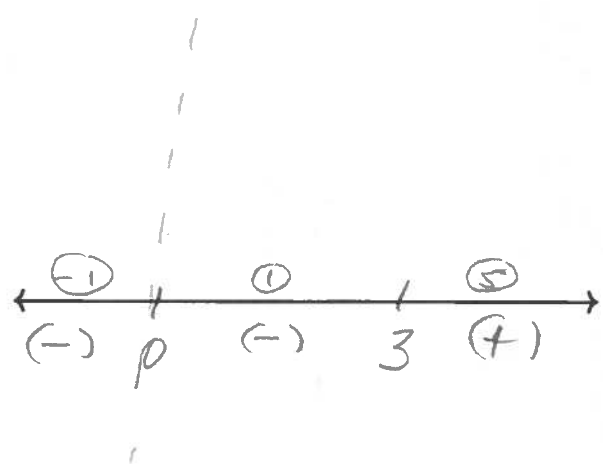
$x = 0$

concave up: $(3, \infty)$

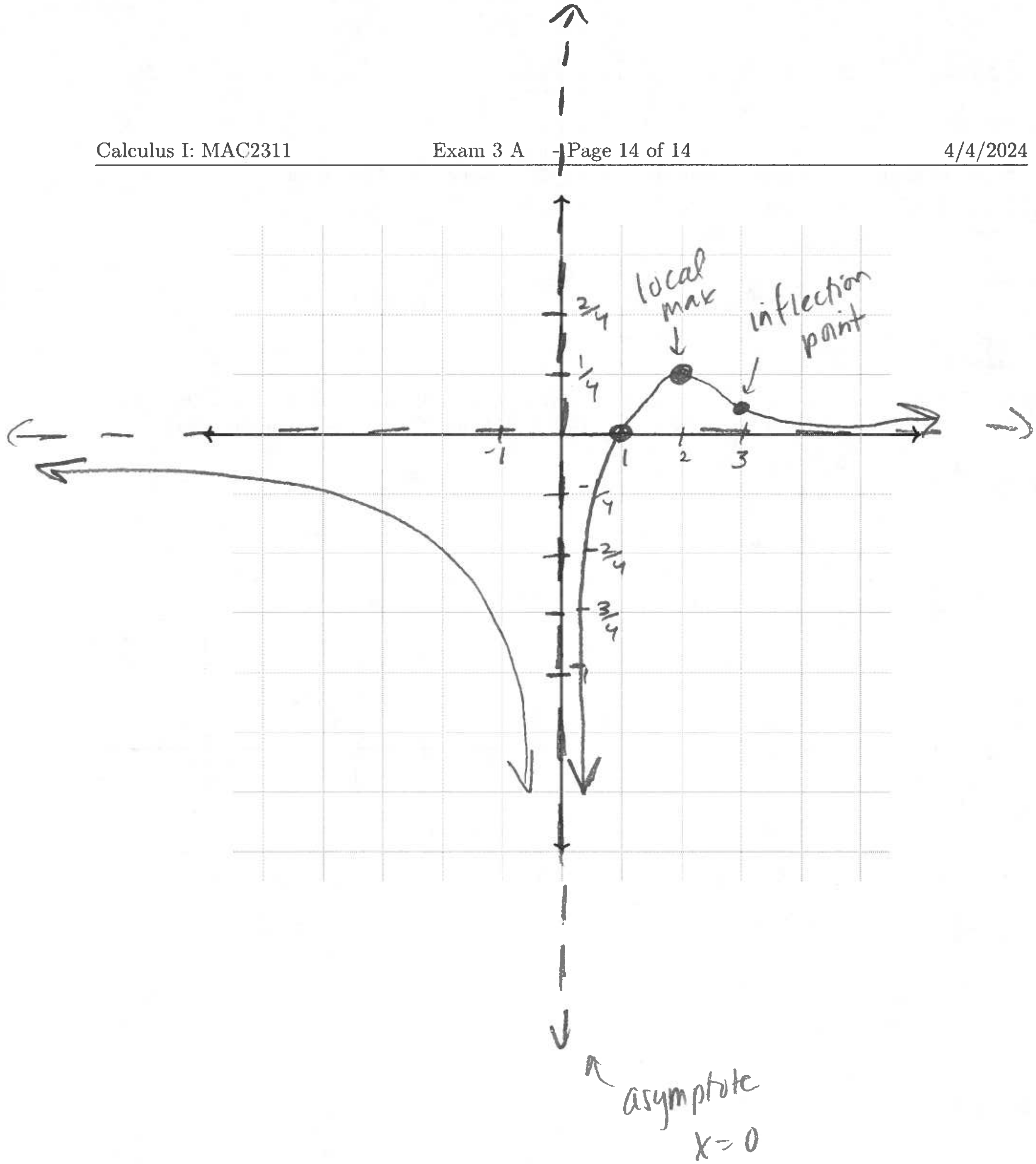
concave down: $(-\infty, 0) \cup (0, 3)$

inflection point: $(3, \frac{2}{9})$

f''



Using the earlier parts, sketch the graph of $f(x) = \frac{x-1}{x^2}$ on the next page.



local max
inflection point
2/4
1/4
-1
1
2
3
-1/4
-2/4
-3/4
-1
asymptote
x = 0