1) Let $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{c} = \vec{j} - 5\vec{k}$. Find the following.

(1) $|\vec{a}|$

Just like distance formula.

$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$

(2) $\vec{a} \cdot \vec{b}$

Multiply components, then add.

$\vec{a} \cdot \vec{b} = 1(3) + 1(-2) + (-2)(1) = 3 - 2 - 2 = -1$

(3) $\vec{a} \times \vec{b}$

Set up matrix with $\vec{i}, \vec{j}, \vec{k}$ and your two vectors

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= \vec{i}\,(1\cdot 1 - (-2)(-2)) - \vec{j}\,(1\cdot 1 - (-2)(3)) + \vec{k}\,(1(-2) - (1)(3))$$

$$= \vec{i}\,(1-4) - \vec{j}\,(1-(-6)) + \vec{k}\,(-2-3)$$

$$= \boxed{\langle -3, -7, -5 \rangle}$$

(4) $\vec{a}\cdot(\vec{b}\times\vec{c})$

Set up a matrix with your three vectors

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= 1(10-1) - 1(-15-0) - 2(3-0)$$

$$= 9 + 15 - 6 = \boxed{18}$$

---

(5) The angle between $\vec{a}$ and $\vec{b}$.

There is a handy property of dot product that says $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$ which we rewrite as $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$.

We know $\vec{a}\cdot\vec{b} = -1$ and $|\vec{a}| = \sqrt{6}$. You can find $|\vec{b}| = \sqrt{14}$, so

$$\cos\theta = \frac{-1}{\sqrt{6}\cdot\sqrt{14}} = \frac{-1}{\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{2}\cdot\sqrt{7}} = \frac{-1}{2\sqrt{21}}$$

so then $\theta = \boxed{\arccos\left(\dfrac{-1}{2\sqrt{21}}\right)}$

(6) The scalar projection of $\vec{b}$ onto $\vec{a}$.

This quantity is written $\text{comp}_{\vec{a}}\vec{b}$ (for "component") and is given by $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

So for us, $\text{comp}_{\vec{a}}\vec{b} = \boxed{\dfrac{-1}{\sqrt{6}}}$

---

(7) The vector projection of $\vec{b}$ onto $\vec{a}$.

This is denoted $\text{proj}_{\vec{a}}\vec{b}$ and is given by

$$\text{proj}_{\vec{a}}\vec{b} = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}}_{\text{most common form}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|} = \left(\text{comp}_{\vec{a}}\vec{b}\right)\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

For us, $\text{proj}_{\vec{a}}\vec{b} = \dfrac{-1}{\sqrt{6}}\left(\dfrac{1}{\sqrt{6}}\langle 1, 1, -2\rangle\right) = \boxed{\dfrac{-1}{6}\langle 1, 1, -2\rangle}$

---

(8) The area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

This area is equal to $|\vec{a}\times\vec{b}|$. We know from (3) that $\vec{a}\times\vec{b} = \langle -3, -7, -5\rangle$ so

$$|\vec{a}\times\vec{b}| = \sqrt{(-3)^2+(-7)^2+(-5)^2} = \sqrt{9+49+25} = \boxed{\sqrt{83}}$$

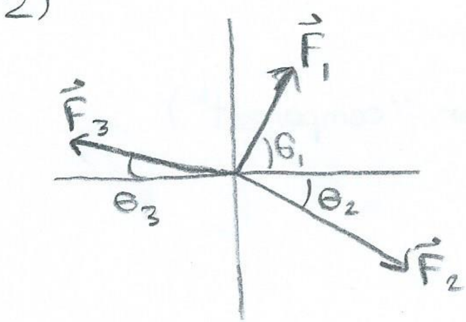
---

(9) The volume of the parallelapiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

This volume is equal to $\vec{a}\cdot(\vec{b}\times\vec{c})$ which we found in (4)

Volume $= \boxed{18}$

2)



The system is in equilibrium, so $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$. We are given

$\theta_1 = \pi/3 \qquad |\vec{F}_1| = 5$

$\theta_2 = \pi/6 \qquad |\vec{F}_2| = 10$

We are asked to find $|\vec{F}_3|$ & θ_3.

$$\vec{F}_1 = \langle |\vec{F}_1|\cos\theta_1, |\vec{F}_1|\sin\theta_1 \rangle = \langle 5\cos(\pi/3), 5\sin(\pi/3) \rangle$$
$$= \left\langle \frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$$

For $\vec{F}_2$, we have to realize that θ_2 is not given in standard form, meaning from the positive x-axis counterclockwise. In standard form, the angle of $\vec{F}_2$ is $-\pi/6$ or $11\pi/6$.

$$\vec{F}_2 = \langle 10\cos(11\pi/6), 10\sin(11\pi/6) \rangle = \left\langle 10 \cdot \frac{\sqrt{3}}{2}, 10 \cdot \left(-\frac{1}{2}\right) \right\rangle$$
$$= \langle 5\sqrt{3}, -5 \rangle.$$

Since $\vec{F}_3 = -\vec{F}_1 - \vec{F}_2$, we find $\vec{F}_3 = \left\langle -\frac{5}{2} - 5\sqrt{3}, -\frac{5\sqrt{3}}{2} + 5 \right\rangle$

So $\vec{F}_3 = \left\langle \frac{-5-10\sqrt{3}}{2}, \frac{-5\sqrt{3}+10}{2} \right\rangle$. If you're careful with the algebra, you find $|\vec{F}_3| = \boxed{5\sqrt{5}\text{ N.}}$

For the angle, $\theta = \arctan(y/x)$, but this gives standard angle, which we can see θ_3 is not standard. So use $|y|$ and $|x|$

$$\theta_3 = \arctan\left(\frac{|-5\sqrt{3}+10|}{|-5-10\sqrt{3}|}\right) = \arctan\left(\frac{10-5\sqrt{3}}{5+10\sqrt{3}}\right)$$

$$= \boxed{\arctan\left(\frac{2-\sqrt{3}}{1+2\sqrt{3}}\right)}$$

3) Three forces $\vec{F}_1=\langle 2,1,1\rangle$, $\vec{F}_2=\langle -1,5,3\rangle$, and $\vec{F}_3$ act on an object. The net force has magnitude 6 and is in the direction of $\langle 1,-2,2\rangle$. Find $\vec{F}_3$.

First we find the net force. $|\langle 1,-2,2\rangle| = 3$ but we are told that the magnitude of the net force is 6. That means the given vector is half our net force since $3/6 = 1/2$. So net force $= \langle 2,-4,4\rangle$. Then

$$\underbrace{\langle 2,1,1\rangle}_{\vec{F}_1}+\underbrace{\langle -1,5,3\rangle}_{\vec{F}_2}+\vec{F}_3=\langle 2,-4,4\rangle$$

$$\langle 1,6,4\rangle+\vec{F}_3=\langle 2,-4,4\rangle$$

$$\vec{F}_3=\boxed{\langle 1,-10,0\rangle}$$

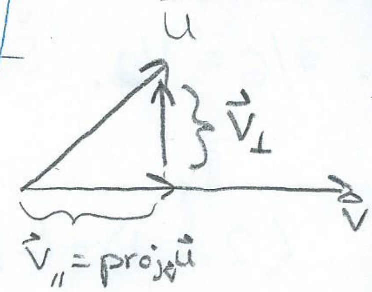
4) Assume that $\vec{u}\cdot\vec{v}=-3$ and $|\vec{v}|=2$. Find $\vec{v}\cdot(2\vec{u}-3\vec{v})$.

$$\begin{aligned}\vec{v}\cdot(2\vec{u}-3\vec{v}) &= 2(\vec{v}\cdot\vec{u})-3(\vec{v}\cdot\vec{v})\\ &=2(\vec{u}\cdot\vec{v})-3|\vec{v}|^2\\ &=2(-3)-3(2^2)\\ &=-6-12\\ &=\boxed{-18}\end{aligned}$$

5) Let $\vec{u} = \langle 3, -1, 2 \rangle$ and $\vec{v} = \langle -2, 1, -1 \rangle$. Express $\vec{u}$ as $\vec{u} = \vec{v}_{\parallel} + \vec{v}_{\perp}$ where $\vec{v}_{\parallel}$ is parallel to $\vec{v}$ and $\vec{v}_{\perp}$ is perpendicular to $\vec{v}$.

Notice that $\text{proj}_{\vec{v}}\vec{u}$ is the vector component of $\vec{u}$ parallel to $\vec{v}$. So then $\vec{v}_{\parallel} = \text{proj}_{\vec{v}}\vec{u}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{-9}{6}\langle -2, 1, -1 \rangle = \boxed{\left\langle 3, \frac{-3}{2}, \frac{3}{2} \right\rangle}$$



Since $\vec{u} = \vec{v}_{\parallel} + \vec{v}_{\perp}$, $\vec{v}_{\perp} = \vec{u} - \vec{v}_{\parallel}$.

$$\vec{v}_{\perp} = \langle 3, -1, 2 \rangle - \left\langle 3, \frac{-3}{2}, \frac{3}{2} \right\rangle = \boxed{\left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle}$$

6) If $A = (1, -2, 3)$, $B = (-1, 4, 5)$, and $C = (0, -1, 3)$ are three points in space, find the following.

(1) The point closest to the xz-plane and the point closest to the plane $x = -2$.

The xz-plane is the plane $y = 0$, so we look for the point with a y-component closest from 0. Clearly it's $\boxed{C}$.

Now we find the point whose x-component is closest from -2. It's point $\boxed{B}$ since the distance between -2 and -1 is 1.

(2) An equation of a sphere with diameter AB.

We need the center of the sphere and its radius.

The length of the diameter is the distance from A to B.

$$d = \sqrt{(-1-1)^2 + (4+2)^2 + (5-3)^2} = \sqrt{(-2)^2 + 6^2 + 2^2} = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$$

and $d = 2r$ so $r = \sqrt{11}$

The center of the sphere is the midpoint of any diameter.

$$C = \left(\frac{1+(-1)}{2}, \frac{-2+4}{2}, \frac{3+5}{2}\right) = (0, 1, 4)$$

$$x^2 + (y-1)^2 + (z-4)^2 = 11$$

(3) A unit vector perpendicular to the plane containing A, B, and C.

We need to find two vectors between the points and cross them.

$\vec{AB} = \langle -2, 6, 2 \rangle$ & $\vec{AC} = \langle -1, 1, 0 \rangle$ so then

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 6 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \langle -2, -2, 4 \rangle$$

To make this a unit vector, we just divide by its magnitude.

$$|\langle -2, -2, 4 \rangle| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

$$\frac{1}{2\sqrt{6}} \langle -2, -2, 4 \rangle = \boxed{\frac{-1}{\sqrt{6}} \langle 1, 1, -2 \rangle}$$

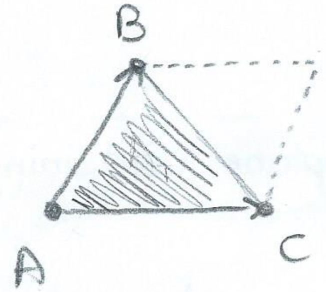
(4) An equation of a plane containing A, B, and C.

From (3) we already found a unit normal vector, so we can use any vector parallel to it. Let's use $\langle 1, 1, -2 \rangle$. We also need any point in the plane, so let's use A.

$$\langle 1, 1, -2 \rangle \cdot \langle x-1, y+2, z-3 \rangle = 0$$

$$x-1+y+2-2z+6=0 \implies \boxed{x+y-2z+7=0}$$

(5) The area of the triangle ABC.

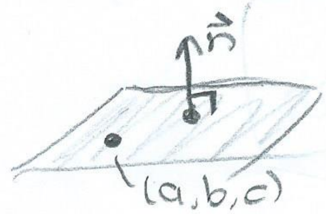


The area of the triangle is half the area of the parallelogram, and that area is given by $|\vec{AB} \times \vec{AC}|$.

We found $\vec{AB} \times \vec{AC} = \langle -2, -2, 4 \rangle$ and we got that its magnitude is $2\sqrt{6}$. So the area of the parallelogram is $2\sqrt{6}$ and the area of the triangle is $\boxed{\sqrt{6}.}$

(*) I forgot to write this in part (4), but a plane is defined by a normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ and a point in the plane (a,b,c) by the equation

$$\langle n_1, n_2, n_3 \rangle \cdot \langle x-a, y-b, z-c \rangle = 0.$$ x, y, z are variables



7)

(1) Determine if $A=(1,0,1)$, $B=(2,-1,3)$, and $C=(3,-2,5)$ lie on the same line.

We can see that if they do lie on the same line, the vectors between them should be parallel.

$\vec{AB}=\langle 1,-1,2\rangle$ and $\vec{AC}=\langle 2,-2,4\rangle$. Clearly $\vec{AC}=2\vec{AB}$ so they are parallel. $\boxed{\text{Then } A, B, C \text{ are colinear.}}$

---

(2) Determine if $P(1,1,1)$, $Q(2,0,3)$, $R(4,1,7)$, and $S(3,-1,-2)$ lie on the same plane.

Four points are enough to construct any parallelepiped. If four points are coplanar, their parallelepiped will fall inside that plane. Since a plane has no volume, neither does anything inside the plane. So we will find the volume of the parallelepiped given by our points.

$\vec{PQ}=\langle 1,-1,2\rangle$; $\vec{PR}=\langle 3,0,6\rangle$; $\vec{PS}=\langle 2,-2,-3\rangle$

$$\text{Volume} = \vec{PQ}\cdot(\vec{PR}\times\vec{PS}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 6 \\ -2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 6 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}$$

$= 12 - 21 - 12 = -21$. Since this isn't zero, the points are $\boxed{\text{not coplanar.}}$

8) Do the lines $\vec{r}_1(t)=\langle 2+t, 1-2t, t+3\rangle$ and $\vec{r}_2(s)=\langle 1-s, s, 2-s\rangle$ intersect? If so, at what point?

We will set the components of the two vectors equal to each other and see if that gives something that makes sense. So we have

$2+t=1-s$
$1-2t=s$
$t+3=2-s$

If we add 1 to both sides of the first equation, we get the third equation, so these two are equivalent.

So we really have $2+t=1-s$ and $1-2t=s$. Plugging in the second to the first gives

$$2+t=1-(1-2t)$$
$$2+t=2t$$
$$2=t$$

and plugging this in to the second gives $s=1-2(2)=-3$.
Indeed, we can then check that $\vec{r}_1(2)=\vec{r}_2(-3)=\langle 4,-3,5\rangle$

9) Let $L_1: \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $L_2: \frac{x+1}{6}=\frac{y-3}{-1}=\frac{z+5}{2}$

Personally I prefer vector form, so we'll convert to that first.

$t=\frac{x-1}{2} \rightarrow 2t=x-1 \rightarrow x=2t+1$
$t=\frac{y-2}{3} \rightarrow 3t=y-2 \rightarrow y=3t+2$ so $L_1: \vec{r}_1(t)=\langle 2t+1, 3t+2, 4t+3\rangle$
$t=\frac{z-3}{4} \rightarrow 4t=z-3 \rightarrow z=4t+3$

Similarly, $L_2: \vec{r}_2(s)=\langle 6s-1, -s+3, 2s-5\rangle$

Now we can begin.

(1) Is $L_1 // L_2$? Do they intersect?

They aren't parallel because their directional vectors $\langle 2,3,4\rangle$ and $\langle 6,-1,2\rangle$ $\boxed{\text{aren't parallel.}}$ Checking intersection is the same as in problem 8.

$$\left.\begin{aligned} 2t+1 &= 6s-1 \\ 3t+2 &= -s+3 \end{aligned}\right\} \quad \text{Solve the first two by elimination or substitution}$$
$$4t+3 = 2s-5$$

$$\begin{aligned} 2t-6s &= -2 \\ 3t+s &= 1 \end{aligned} \Rightarrow \begin{aligned} &2t-6s=-2 \\ +&(18t+6s=6) \\ \hline &20t=4 \\ &t=1/5 \end{aligned}$$

$$3(1/5)+s=1 \Rightarrow s=2/5$$

So does $t=1/5$ & $s=2/5$ solve the third equation?

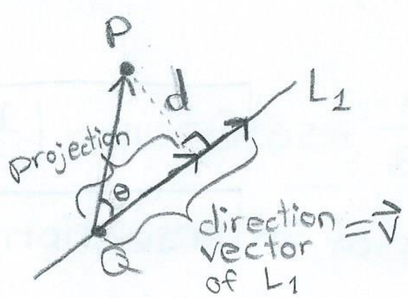
$$4(1/5)+3 \stackrel{?}{=} 2(2/5)-5 \Rightarrow 4/5+3 \stackrel{?}{=} 4/5-5 \Rightarrow 3 \stackrel{?}{=} -5$$

No it doesn't. So the lines $\boxed{\text{don't intersect.}}$

---

(2) Find the distance from the point $(1,1,1)$ to L_1.

Here's a drawing



$d=|\overrightarrow{QP}|\sin\theta$ should be clear from the picture. Also, cross product has the handy property that $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

So for us that becomes $|\overrightarrow{QP}\times\vec{v}|=|\overrightarrow{QP}||\vec{v}|\sin\theta$

or $\dfrac{|\overrightarrow{QP}\times\vec{v}|}{|\vec{v}|}=|\overrightarrow{QP}|\sin\theta=d$

Q can be any point on L_1.

So let's choose $Q=(1,2,3)$. Then $\overrightarrow{QP}=\langle 0,-1,-2\rangle$
and $\vec{v}=\langle 2,3,4\rangle$ from $\vec{r}_1$.

So $\overrightarrow{QP}\times\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -2 \\ 2 & 3 & 4 \end{vmatrix}=\langle 2,-4,2\rangle$

So $|\overrightarrow{QP}\times\vec{v}|=\sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$

And $|\vec{v}|=\sqrt{4+9+16}=\sqrt{29}$

So $d=\frac{2\sqrt{6}}{\sqrt{29}}=\boxed{\sqrt{\frac{24}{29}}}$

10) Let $P_1: x+y-z=1$ and $P_2: x-y-z=5$.

(1) Do the planes intersect?

$\vec{n}_1=\langle 1,1,-1\rangle$ & $\vec{n}_2=\langle 1,-1,-1\rangle$ are not parallel, so $\boxed{\text{yes}}$

(2) Find the angle between P_1 & P_2.

This is the same as the angle between their normal vectors.

$\vec{n}_1\cdot\vec{n}_2=|\vec{n}_1||\vec{n}_2|\cos\theta \Rightarrow \cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

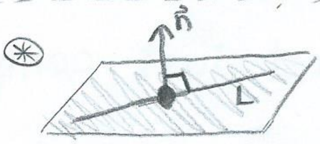
$\vec{n}_1\cdot\vec{n}_2=1(1)+1(-1)+(-1)(-1)=1$

$|\vec{n}_1|=\sqrt{3}$ & $|\vec{n}_2|=\sqrt{3}$ so $\cos\theta=\frac{1}{3}$ so $\boxed{\theta=\arccos\left(\frac{1}{3}\right)}$

(3) Find symmetric equations of the line of intersection.

We'll need the line's directional vector $\vec{v}$ and a point on the line P.

Since $\vec{v}$ is on the line and the line is in each plane, $\vec{v} \perp \vec{n}_1$ and $\vec{v} \perp \vec{n}_2$.⊛ That means $\vec{n}_1 \times \vec{n}_2 \parallel \vec{v}$

⊛ 

This is happening in both planes.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \langle -2, 0, -2 \rangle$$

We only need a vector parallel to this, so let's use $\langle 1,0,1 \rangle$.

Now we need a point P. We can find one by using elimination on the plane equations.

$$\begin{array}{r} x+y-z=1 \\ +(x-y-z=5) \\ \hline 2x-2z=6 \\ x-z=3 \end{array}$$

Now just pick any x & z satisfying $x-z=3$. I like $x=3$ and $z=0$. Plugging these into either plane gives $y=-2$

So $P=(3,-2,0)$

$$\underbrace{\vec{r}(t)}_{y} = \underbrace{t}_{x}\underbrace{\langle 1,0,1 \rangle}_{m} + \underbrace{\langle 3,-2,0 \rangle}_{b} = \boxed{\langle t+3, -2, t \rangle}$$ vector form

$x=t+3 \longrightarrow t=x-3$

$y=-2 \qquad y=-2$

$z=t \qquad t=z$

So $\boxed{z=x-3\,;\, y=-2}$ is a solution.

symmetric form

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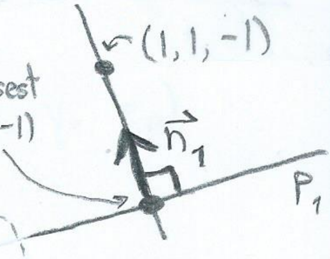
(4) Find the distance from $(1,1,-1)$ to P_1.



$\vec{r}(t) = \langle 1,1,-1 \rangle t + \langle 1,1,-1 \rangle$

$= \langle t+1, t+1, -t-1 \rangle$
$\quad\; x \quad\;\; y \quad\;\; z$

$\Downarrow$

$(t+1)+(t+1)-(-t-1)=1$

$3t+3=1$

$3t=-2$

$t=-2/3$

$\Uparrow$

$r(-2/3) = \langle 1/3, 1/3, -1/3 \rangle$

$\Downarrow$

$d = \sqrt{(1-1/3)^2 + (1-1/3)^2 + (-1+1/3)^2}$

$= \sqrt{(2/3)^2 + (2/3)^2 + (-2/3)^2}$

$= \boxed{2\sqrt{3}/3}$

11) Discuss traces of the surface $x^2-y^2+4z^2+2y=1$ and identify the surface.

First just complete the square for y

$$x^2-y^2+2y+4z^2=1 \Rightarrow x^2-(y^2-2y)+4z^2=1$$

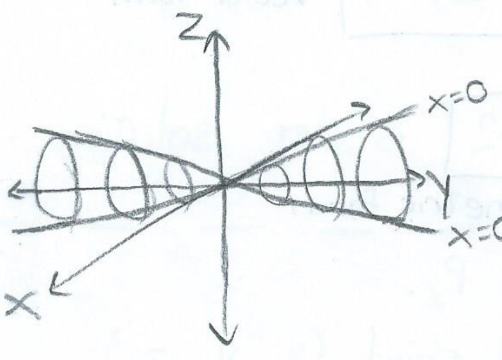
$$\Rightarrow x^2-(y^2-2y+1)+1+4z^2=1 \Rightarrow x^2-(y-1)^2+4z^2=0$$

$$\Rightarrow x^2+4z^2=(y-1)^2$$

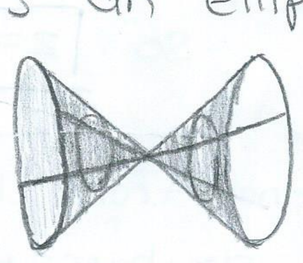
If we take a trace where $y=$constant, we get $x^2+4z^2=k$ which is an ellipse.

If we say $z=0$ we get $x^2=(y-1)^2$ which becomes $x=y-1$ and $-x=y-1$, which are lines

If we say $x=0$, we get the lines $2z=y-1$ & $-2z=y-1$.



It's an elliptical $\boxed{\text{cone.}}$



* Also for $z=$constant and $x=$constant but not zero, we get hyperbolas. I just don't personally find this helpful to find the entire surface, but we are asked to "discuss" as well.

12) Find an equation for the set of all points $P(x,y,z)$ that are equidistant from P to the z-axis and from P to the plane $x=-1$. Identify the surface

Distance from z-axis to P is $d_z = \sqrt{x^2+y^2}$.

Distance from P to plane $x+1=0$ is given by

$$D = \frac{1(x)+0(y)+0(z)+1}{\sqrt{1^2+0^2+0^2}} = \frac{x+1}{\sqrt{1}} = x+1$$

⊛ Generically, for a plane $ax+by+cz+d=0$ and a point (x_0, y_0, z_0)

$$D = \frac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}$$

So $D = d_z \Rightarrow \sqrt{x^2+y^2} = x+1$

$$\Rightarrow x^2+y^2 = (x+1)^2$$

$$\Rightarrow \cancel{x^2}+y^2 = \cancel{x^2}+2x+1$$

$$\Rightarrow \boxed{y^2 = 2x+1}$$ which is a parabolic cylinder

13) Consider the curve $\vec{r}(t) = \langle \cos t, t, -\sin t \rangle$. Find the following.

(1) Find the unit tangent vector $\hat{T}(t)$ and unit normal vector $\hat{N}(t)$:

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad \text{and} \quad \hat{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\vec{r}\,''(t)}{|\vec{r}\,''(t)|}$$

$$\vec{r}\,'(t) = \langle -\sin(t), 1, -\cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \sqrt{2}$$

So $\hat{T}(t) = \boxed{\frac{1}{\sqrt{2}}\langle -\sin t, 1, -\cos t \rangle}$

To find $\hat{N}(t)$, let's use $\hat{N}(t)=\frac{\vec{r}''}{|\vec{r}''|}$.

Since $\vec{r}''(t)=\langle -\cos t, 0, \sin t\rangle$ and $|\vec{r}''(t)|=\sqrt{\cos^2 t+\sin^2 t}=1$

we get $\hat{N}(t)=\boxed{\langle -\cos t, 0, \sin t\rangle}$

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(2) The tangent line to the curve at $(1,0,0)$.

The t that gives $\vec{r}(t)=\langle 1,0,0\rangle$ is $t=0$.

$\vec{r}'(0)=\langle -\sin(0), 1, -\cos(0)\rangle=\langle 0,1,-1\rangle$

So $\vec{L}(s)=s\langle 0,1,-1\rangle+\langle 1,0,0\rangle=\langle 1,s,-s\rangle$ (L for line)

You could also write $\boxed{x=1,\ y=-z}$

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(3) The arc length from $(1,0,0)$ to $(1,2\pi,0)$

$s=\int_{t_1}^{t_2}|\vec{r}'(t)|dt$. The t for $(1,0,0)$ is $t=0$ and for $(1,2\pi,0)$ is $t=2\pi$.

So $s=\int_0^{2\pi}|\vec{r}'(t)|dt=\int_0^{2\pi}\sqrt{2}\,dt=\boxed{2\pi\sqrt{2}}$

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(4) $\frac{ds}{dt}$

$\underbrace{\frac{ds}{dt}=|\vec{r}'(t)|}$ so $\frac{ds}{dt}=\sqrt{2}$

This is a helpful fact to know.

(5) The curvature of the curve at the point $(1,0,0)$.

In general, $K(t) = \dfrac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$.

We have $|\vec{r}'(t)| = \sqrt{2}$ and since $\hat{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin t, 1, -\cos t\rangle$,

$\hat{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, 0, \sin t\rangle$ so $|\hat{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$.

So then $K = \dfrac{1/\sqrt{2}}{\sqrt{2}} = \dfrac{1}{2}$ (everywhere, since we didn't use $(1,0,0)$).

14) Find the curvature of the function $y = x^4$ at $(1,1)$.

For a function, $K = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$.

So $f'(x) = 4x^3$ and $f''(x) = 12x^2$

$$K(x) = \frac{12x^2}{(1+(4x^3)^2)^{3/2}} = \frac{12x^2}{(1+16x^6)^{3/2}}$$

$$K(1) = \frac{12}{17^{3/2}}$$

15) Let $\vec{r}(t) = \langle t^2, 2t, \ln(t) \rangle$ be a vector function describing the path of a particle. Find tangential and normal components of acceleration at $t = 1/2$.

$a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$ & $a_N = \sqrt{|\vec{a}|^2 - a_T^2}$ $\quad (\vec{v} = \vec{r}\,' \text{ \& } \vec{a} = \vec{r}\,'')$

$\vec{v} = \vec{r}\,' = \langle 2t, 2, 1/t \rangle$ so at $t = 1/2$, $\vec{v} = \langle 1, 2, 2 \rangle$

so $|\vec{v}| = \sqrt{1+4+4} = 3$

$\vec{a} = \vec{r}\,'' = \langle 2, 0, -1/t^2 \rangle$ so at $t = 1/2$, $\vec{a} = \langle 2, 0, -4 \rangle$

so $|\vec{a}| = \sqrt{4+16} = \sqrt{20}$

$a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{2+0-8}{3} = \frac{-6}{3} = -2$ at $t = 1/2$

$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{20-4} = \sqrt{16} = 4$

16) For the curve given by $\vec{r}(t)=\langle \sin^3 t, \cos^3 t, \sin^2 t\rangle$, $0<t<\pi/2$ find the following

(1) The unit tangent vector.

$$\vec{r}\,'(t)=\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\rangle$$

$$=\sin t\cos t\,\langle 3\sin t, -3\cos t, 2\rangle$$

$$|\vec{r}\,'(t)| = \sin t\cos t\sqrt{9\sin^2 t+9\cos^2 t+4} = \sin t\cos t\sqrt{13}$$

$$\hat{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\frac{1}{\sqrt{13}\sin t\cos t}\cdot\sin t\cos t\,\langle 3\sin t, -3\cos t, 2\rangle$$

$$=\boxed{\frac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle}$$

(2) The unit normal vector.

$$\hat{T}'(t)=\frac{1}{\sqrt{13}}\langle 3\cos t, 3\sin t, 0\rangle$$

$$|\hat{T}'(t)|=\frac{1}{\sqrt{13}}\sqrt{9\cos^2 t+9\sin^2 t+0}=\frac{3}{\sqrt{13}}$$

$$\hat{N}(t)=\frac{\hat{T}'(t)}{|\hat{T}'(t)|}=\frac{\sqrt{13}}{3}\cdot\frac{1}{\sqrt{13}}\langle 3\cos t, 3\sin t, 0\rangle=\boxed{\langle \cos t, \sin t, 0\rangle}$$

(3) The binormal vector.

By definition, $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$

So $\hat{B}(t) = \frac{1}{\sqrt{13}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\sin t & -3\cos t & 2 \\ \cos t & \sin t & 0 \end{vmatrix} = \boxed{\frac{1}{\sqrt{13}} \langle -2\sin t, 2\cos t, 3 \rangle}$

(4) The curvature.

$$K = \frac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \frac{3/\sqrt{13}}{\sqrt{13}\sin t\cos t} = \boxed{\frac{3}{13\sin t\cos t}}$$

17) Let $\vec{r}(t) = \langle t\ln(t), \sin(\pi t), \sqrt{5-t} \rangle$ be a vector function

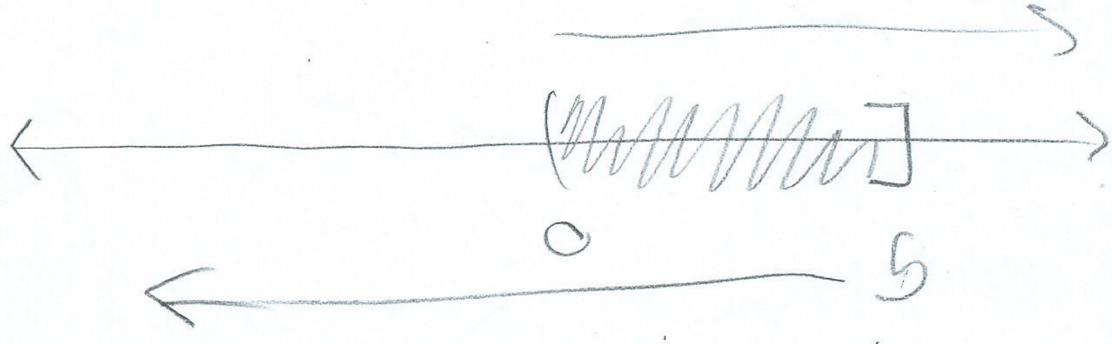
(1) Find the domain of $\vec{r}(t)$.

Domain of $t\ln(t)$ is $(0, \infty)$

Domain of $\sin(\pi t)$ is $(-\infty, \infty)$

Domain of $\sqrt{5-t}$ is $5-t \geq 0 \Rightarrow t \leq 5 \Rightarrow (-\infty, 5]$.

So the domain of $\vec{r}(t)$ is $(0, 5]$



(2) Find $\lim_{t\to 0^+} \vec{r}(t)$

We just take the limit of each component.

$\lim_{t\to 0^+} \sin(\pi t) = \sin(0) = 0$ & $\lim_{t\to 0^+} \sqrt{5-t} = \sqrt{5}$

$\lim_{t\to 0^+} t\ln(t)$ "=" $0 \cdot -\infty$

$$= \lim_{t\to 0^+} \frac{\ln(t)}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0$$

So $\lim_{t\to 0^+} \vec{r}(t) = \langle 0, 0, \sqrt{5} \rangle$

(3) Find $\int \vec{r}(t)\,dt$.

$$\int t\ln t\,dt = \frac{t^2}{2}\ln t - \int \frac{t^2}{2}\cdot\frac{1}{t}\,dt = \frac{t^2\ln t}{2} - \int \frac{t}{2}\,dt = \frac{t^2\ln t}{2} - \frac{t^2}{4}$$

$u = \ln t$ $\quad v = t^2/2$
$du = 1/t\,dt$ $\quad dv = t\,dt$

$$\int \sin(\pi t)\,dt = -\frac{\cos(\pi t)}{\pi}$$

$$\int \sqrt{5-t}\,dt = \frac{-2}{3}(5-t)^{3/2}$$

so $\int \vec{r}(t)\,dt = \left\langle \frac{t^2}{2}\ln t - \frac{t^2}{4}, \frac{-\cos(\pi t)}{\pi}, \frac{-2}{3}(5-t)^{3/2} \right\rangle + \vec{C}$

(4) Let C be the curve parameterized by $\vec{r}(t)$. Find a and b if the vector $\langle a, b, 1\rangle$ is parallel to the tangent vector of the curve at the point $(0,0,2)$.

To find t, use any coordinate.

$\sqrt{5-t} = 2 \Rightarrow 5-t=4 \Rightarrow t=1$

$\vec{r}'(t)$ is tangent to C.

$\vec{r}'(t) = \left\langle \ln t + 1, \pi\cos(\pi t), \frac{-1}{2\sqrt{5-x}} \right\rangle$

$\vec{r}'(1) = \left\langle 1, -\pi, \frac{-1}{4} \right\rangle$

Since this and $\langle a,b,1\rangle$ are parallel, $\langle a,b,1\rangle = k\langle 1,-\pi,\frac{-1}{4}\rangle$ for some constant k. So $1 = -\frac{k}{4}$ so $k=-4$

$a = -4\cdot 1 = -4$

$b = -4\cdot -\pi = 4\pi$