MAC 2312 Exam 3 - Form A April 8, 2024

Name: Solutions **Section #:** ______________

UF-ID: ______________________ **TA Name:** ______________

A: Sign the back of your scantron sheet.

B: On the indicated spaces on the front of your scantron, **in pencil**, write and encode:

1. Your name (last name, first initial, middle initial)
2. Your UFID Number
3. Your 4-digit Section Number

C: Under "special codes", code in the test ID number 3,1:

1 2 • 4 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, for "Test Form Code", encode A .

• B C D E

E: Some basic information about the exam:

1. This exam has two parts: a 10-question multiple choice section worth 50 points, and a 4-question free response section worth 55 points. The entire exam is out of 100 points.
2. You will have 100 minutes to take the exam.
3. You may write on your exam.
4. Raise your hand if you need more scratch paper or if you have a problem with the test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE EXAM.

F: KEEP YOUR SCANTRON COVERED AT ALL TIMES

G: When you are finished:

1. Before turning in your test, check for transcribing errors. Any mistakes you leave in are there to stay.
2. Turn in your scantron and the free-response portion of your exam to your TA, or the proctor designated as your TA. Be prepared to show your UF-ID card.
3. Solutions to the exam will be posted on Canvas after the exam is over.

The Honor Pledge: "On my honor, I have neither given nor received unauthorized aid doing this exam."

Signature: ______________________________

Part I: Multiple Choice There are **10** questions on this portion of the exam. Fill in the answers to these questions on the provided scantron sheet. Only answers on the scantron will be graded. Each problem is worth 5 points, for a total of 50 points on this portion of the exam.

1. The power series representation for $f(x) = \dfrac{x^2}{x+4}$ centered at $a = 0$ is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^{n-3}}$.

(A) True

(B) False

$$\frac{x^2}{x+4} = x^2 \cdot \frac{1}{4+x} = x^2 \cdot \frac{1}{4(1+x/4)} = \frac{x^2}{4} \cdot \frac{1}{1-(-x/4)}$$

$$= \frac{x^2}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$$

$$= \frac{x^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^{n-1}}$$

2. Use the Alternating Series Estimation Theorem to find the maximum possible error of using the first three nonzero terms of the Maclaurin series for $f(x) = \cos x$ to approximate $\cos\sqrt{2}$.

(A) $\frac{1}{720}$ (B) $\frac{1}{360}$ (C) $\frac{1}{240}$ (D) $\frac{1}{120}$ (E) $\frac{1}{90}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \boxed{\frac{x^6}{6!}} \dots$$

$$\text{Maximum error} = \frac{(\sqrt{2})^6}{6!} = \frac{2^3}{6!} = \frac{8}{720} = \frac{1}{90}$$

3. Let $f(x) = x\cos(3x)$. Which of the following is the Maclaurin series for $f(x)$?

(A) $3 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \ldots$

(B) $3x - \frac{9}{2}x^3 + \frac{27}{8}x^5 + \ldots$

(C) $x - \frac{3}{2}x^3 + \frac{9}{8}x^5 + \ldots$

(D) $x - \frac{9}{2}x^3 + \frac{27}{8}x^5 + \ldots$

(E) $x + \frac{9}{2}x^3 + \frac{27}{8}x^5 + \ldots$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}$$

$$\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{2n}}{(2n)!}$$

$$\frac{27x^3}{8}$$

$$x\cos(3x) = x \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{2n+1}}{(2n)!} = x - \frac{9x^3}{2} + \frac{81x^3}{24} - \ldots$$

4. Let $f(x) = x^7 \arctan(3x^2)$. Evaluate $f^{(13)}(0)$, the 13th derivative of $f(x)$ when $x = 0$.

(A) $3 \cdot 13!$ (B) $-9 \cdot 13!$ (C) $27 \cdot 13!$ (D) $-81 \cdot 13!$ (E) $243 \cdot 13!$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow x^7 \arctan(3x^2) = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1} \cdot x^{4n+9}}{2n+1}$$

$f^{(13)}(0) = C_{13} \cdot 13!$ $\quad$ C_{13} occurs when $4n+9 = 13$

$\Rightarrow n = 1$

$$f^{(13)}(0) = \frac{(-1)^1 \cdot 3^3}{3} \cdot 13! = -9 \cdot 13!$$

5. Use the first 2 terms of the power series representation for $\ln(1-x^3)$ centered at $a=0$ to approximate $\ln\left(\frac{7}{8}\right)$.

(A) $-\frac{2}{3}$ (B) $-\frac{3}{2}$ (C) $-\frac{17}{128}$ (D) $-\frac{369}{512}$ (E) $-\frac{119}{512}$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \Rightarrow \ln(1-x^3) = -\sum_{n=1}^{\infty} \frac{(x^3)^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{3n}}{n}$$

$$\ln 7/8 = \ln(1-(1/2)^3) = -\sum_{n=1}^{\infty} \frac{(1/2)^{3n}}{n}$$

$$\approx -\left(\frac{1}{2}\right)^3 - \frac{\left(\frac{1}{2}\right)^6}{2} = -\frac{1}{2^3} - \frac{1}{2^7} = -\frac{2^4+1}{2^7}$$

$$= -\frac{17}{128}$$

6. What is the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-4)^n}{(-3)^{n+2}(n+2)}$?

(A) $\left[\frac{11}{3}, \frac{13}{3}\right]$ (B) $\left[\frac{11}{3}, \frac{13}{3}\right)$ (C) $(1,7)$ (D) $(1,7]$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(-3)^{n+3}(n+3)} \cdot \frac{(-3)^{n+2}(n+2)}{(x-4)^n}\right|$$

center

$$= \lim_{n\to\infty} \frac{n+2}{3(n+3)} |x-4| = \frac{|x-4|}{3} < 1 \Rightarrow |x-4| < 3 \Leftarrow \text{ROC}$$

$x=1$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{(-3)^{n+2}(n+2)} = \sum_{n=0}^{\infty} \frac{1}{9(n+2)}$ diverges by LCT

$x=7$: $\sum_{n=0}^{\infty} \frac{3^n}{(-3)^{n+2}(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{9(n+2)}$ converges by AST $\Rightarrow$ IOC: $(1,7]$

7. The second degree Taylor polynomial for $f(x) = \frac{3}{x+1}$ centeted at $a = 1$ is

$$T_2(x) = \frac{3}{2} - \frac{3}{4}(x-1) + C(x-1)^2.$$

What is the value of C?

(A) $-\frac{3}{8}$ (B) $-\frac{3}{4}$ (C) $\frac{1}{8}$ (D) $\frac{3}{4}$ (E) $\frac{3}{8}$

$$f'(x) = \frac{-3}{(x+1)^2} \Rightarrow f''(x) = \frac{6}{(x+1)^3}$$

$$f''(1) = \frac{6}{2^3} = \frac{6}{8} = \frac{3}{4}$$

$$C = \frac{3/4}{2!} = 3/8$$

8. Evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$.

(A) e (B) $e-1$ (C) $\frac{1}{e} - \frac{\sqrt{2}}{2}$ (D) $\frac{1}{e} - 1$ (E) $\pi + \frac{1}{2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} = e^1 = e$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin \pi = 0$$

$$e - 0 = e$$

9. Evaluate $\lim_{x \to 0} \dfrac{\sin(x^3) - x^3}{x^9}$

(A) 1 (B) 0 (C) $-\frac{1}{6}$ (D) $\frac{1}{6}$ (E) $\frac{1}{120}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots$$

$$\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x \to 0} \frac{-\frac{x^9}{6} + \frac{x^{15}}{120} - \dots}{x^9}$$

$$= \lim_{x \to 0} \left(-\frac{1}{6} + \frac{x^6}{120} - \dots\right) = -\frac{1}{6}$$

10. Let $f(x)$ be a function with continuous derivatives such that $|f^{(5)}(c)| \leq 3$ for all $0 \leq c \leq 2$. What is the maximum possible error of using the 4th degree Maclaurin series of $f(x)$ to approximate $f(2)$?

(A) $\frac{2}{5}$ (B) $\frac{8}{15}$ (C) $\frac{2}{3}$ (D) $\frac{4}{5}$ (E) $\frac{14}{15}$

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

$$|R_4(2)| \leq \frac{3|2-0|^5}{5!} = \frac{3 \cdot 32}{120} = \frac{32}{40} = \frac{4}{5}$$

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UF-ID: ____________________ **TA Name:** __________

Part II: Free Response There are **4** questions on this portion of the exam. Show **ALL** work clearly in the space provided for each problem, unless the problem says otherwise. Your work must be complete, logical, and understandable or it will receive no credit. Please cross out or fully erase any work that you do not want graded. A total of **55** points are available on this portion of the exam.

FR Scores	
1	/15
2	/15
3	/10
4	/15
FR Total	/55

1(a). Find the third degree Taylor polynomial for $f(x) = \sin x$ centered at $a = \frac{\pi}{6}$.

$f(x) = \sin x$ $\quad f(\pi/6) = \sin \pi/6 = 1/2$ $\quad C_0 = \frac{1/2}{0!} = 1/2$

$f'(x) = \cos x$ $\quad f'(\pi/6) = \cos \pi/6 = \frac{\sqrt{3}}{2}$ $\quad C_1 = \frac{\sqrt{3}/2}{1!} = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\quad f''(\pi/6) = -\sin \pi/6 = -\frac{1}{2}$ $\quad C_2 = \frac{-1/2}{2!} = -\frac{1}{4}$

$f'''(x) = -\cos x$ $\quad f'''(\pi/6) = -\cos \pi/6 = -\sqrt{3}/2$ $\quad C_3 = \frac{-\sqrt{3}/2}{3!} = \frac{-\sqrt{3}}{12}$

$$\boxed{T_3 = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{4}(x - \frac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x - \frac{\pi}{6})^3}$$

(b). Explain why you cannot use the Alternating Series Estimation Theorem to find the maximum possible error of using your answer from part (a) to approximate sin 1.

We cannot use the Alternating Series Estimation Theorem because the series is not an alternating series.

(c). Use Taylor's Remainder Estimation Theorem to find the maximum possible error of using your answer from part (a) to approximate sin 1. Use the fact that $|\sin x| \le 1$ for all real numbers x.

$f^{(4)}(x) = \sin x \Rightarrow |f^{(4)}(x)| = |\sin x| \le 1 = M$ for all x

$$|R_3(x)| \le \frac{M|x-a|^{3+1}}{(3+1)!} = \frac{1 \cdot (1 - \pi/6)^4}{4!}$$

$$= \frac{(1 - \pi/6)^4}{24}$$

2(a). Find the power series representation for $f(x) = \dfrac{1}{6-x}$ centered at $a = 0$.

$$\frac{1}{6-x} = \frac{1}{6(1-x/6)} = \frac{1}{6}\sum_{n=0}^{\infty} (x/6)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}}}$$

(b). Use your answer from part (a) to find the power series representation for $g(x) = \dfrac{1}{(6-x)^2}$ centered at $a = 0$.

$$\frac{1}{(6-x)^2} = \frac{d}{dx}\left[\frac{1}{6-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}}\right]$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{n x^{n-1}}{6^{n+1}}}$$

(c). Use your answer from part (b) to find the power series representation for $h(x) = \dfrac{x^2}{(6-x)^2}$ centered at $a = 0$.

$$\frac{x^2}{(6-x)^2} = x^2 \cdot \frac{1}{(6-x)^2} = x^2 \sum_{n=1}^{\infty} \frac{n x^{n-1}}{6^{n+1}} = \boxed{\sum_{n=1}^{\infty} \frac{n x^{n+1}}{6^{n+1}}}$$

3(a). Find the power series representation for $f(x) = \dfrac{1}{x^2 + 4x + 6}$ centered at $a = -2$.

$$\frac{1}{x^2+4x+6} = \frac{1}{x^2+4x+4-4+6}$$

$$= \frac{1}{2+(x+2)^2}$$

$$= \frac{1}{2\left(1-\frac{-(x+2)^2}{2}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-(x+2)^2}{2}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(x+2)^{2n}}{2^{n+1}}}$$

(b). Find the interval of convergence of your answer from part (a). Justify your answer.

Because this series is a geometric series, it converges when $|r| = \left|\frac{-(x+2)^2}{2}\right| < 1$

$$(x+2)^2 < 2$$

$$|x+2| < \sqrt{2}$$

IOC: $\boxed{(-2-\sqrt{2}, -2+\sqrt{2})}$

4(a). Find the Maclaurin series for $f(x) = \ln(4x^2 + 1)$.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \Rightarrow \ln(4x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(4x^2)^n}{n}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n x^{2n}}{n}}$$

(b). Use the first two nonzero terms of your answer from part (a) to approximate $\int_0^{1/2} \ln(4x^2+1)\,dx$.

You may assume $x = 1/2$ is in the interval of convergence of the Maclaurin series for $\int \ln(4x^2+1)\,dx$.

$$\int \ln(4x^2+1)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n x^{2n}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n x^{2n+1}}{n(2n+1)}$$

$$= \frac{4}{3}x^3 - \frac{8}{5}x^5 + \frac{64x^7}{21} - \dots$$

$$\int_0^{1/2} \ln(4x^2+1)\,dx \approx \left[\frac{4}{3}x^3 - \frac{8}{5}x^5\right]_0^{1/2} = \frac{4}{3}\cdot\frac{1}{8} - \frac{8}{5}\cdot\frac{1}{32} = \frac{1}{6} - \frac{1}{20}$$

$$= \boxed{7/60}$$

(c). Use the Alternating Series Estimation Theorem to find the maximum possible error of your approximation from part (b).

$$\text{Maximum error} = \frac{64x^7}{21}\Big|_0^{1/2} = \frac{64}{21}\cdot\frac{1}{128} = \boxed{\frac{1}{42}}$$

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$f(x)$	Maclaurin Series	Interval of Convergence
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \dfrac{x^n}{n}$	$[-1,1)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{x^n}{n}$	$(-1,1]$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$(-\infty,\infty)$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n\dfrac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n\dfrac{x^{2n+1}}{2n+1}$	$[-1,1]$

Taylor Remainder Estimate:

If $\left|f^{(N+1)}(c)\right| \le M$ for all numbers c between x and a, then the remainder $R_N(x)$ of the Taylor series satisfies the inequality

$$|R_N(x)| \le \frac{M|x-a|^{N+1}}{(N+1)!} \quad \text{for all numbers between } x \text{ and } a.$$

Taylor Coefficients:

$$c_n = \frac{f^{(n)}(a)}{n!}$$