Name: Solutions **Section #:** ________

UF-ID: ________ **TA Name:** ________

A: Sign the back of your scantron sheet.

B: On the indicated spaces on the front of your scantron, **in pencil**, write and encode:

1. Your name (last name, first initial, middle initial)
2. Your UFID Number
3. Your 4-digit Section Number

C: Under "special codes", code in the test ID number 2,1:

1 • 3 4 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, for "Test Form Code", encode A .

• B C D E

E: Some basic information about the exam:

1. This exam has two parts: a 10-question multiple choice section worth 50 points, and a 4-question free response section worth 55 points. The entire exam is out of 100 points.

2. You will have 100 minutes to take the exam.

3. You may write on your exam.

4. Raise your hand if you need more scratch paper or if you have a problem with the test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE EXAM.

F: KEEP YOUR SCANTRON COVERED AT ALL TIMES

G: When you are finished:

1. Before turning in your test, check for transcribing errors. Any mistakes you leave in are there to stay.

2. Turn in your scantron and the free-response portion of your exam to your TA, or the proctor designated as your TA. Be prepared to show your UF-ID card.

3. Solutions to the exam will be posted on Canvas after the exam is over.

The Honor Pledge: "On my honor, I have neither given nor received unauthorized aid doing this exam."

Signature: ________

Part I: Multiple Choice There are **10** questions on this portion of the exam. Fill in the answers to these questions on the provided scantron sheet. Only answers on the scantron will be graded. Each problem is worth 5 points, for a total of 50 points on this portion of the exam.

1. Which of the following sequences converge?

$$a_n = \frac{(-2)^{n+1}}{3^n} \qquad b_n = \frac{(\ln n)^3}{\sqrt[3]{n}} \qquad c_n = \frac{5 - \sin n}{n} \qquad d_n = n \sin(\pi/n)$$

(A) a_n, b_n, and c_n only

(B) a_n, b_n, and d_n only

(C) a_n, c_n, and d_n only

(D) b_n, c_n, and d_n only

(E) a_n, b_n, c_n, and d_n

$\lim_{n\to\infty} \frac{(-2)^{n+1}}{3^n} = 0 \Rightarrow \{a_n\}$ converges

$\lim_{n\to\infty} \frac{(\ln n)^3}{\sqrt[3]{n}} = 0 \Rightarrow \{b_n\}$ converges

$\lim_{n\to\infty} \frac{5-\sin n}{n} = 0 \Rightarrow \{c_n\}$ converges

$\lim_{n\to\infty} n\sin(\pi/n) = \lim_{n\to\infty} \pi \cdot \frac{\sin(\pi/n)}{\pi/n} = \pi \Rightarrow \{d_n\}$ converges

2. Which of the following statements is true concerning the series $\sum_{n=1}^{\infty} \frac{n}{n^3+5}$?

(A) Converges by the Direct Comparison Test

(B) Diverges by the Direct Comparison Test

(C) Converges by the Test for Divergence

(D) Diverges by the Test for Divergence

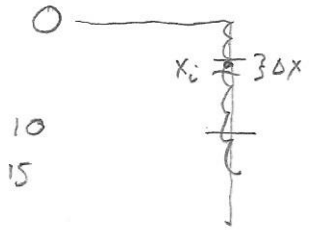
(E) Converges by the Ratio Test

$a_n = \frac{n}{n^3+5} < \frac{n}{n^3} = \frac{1}{n^2} = b_n$

Because $\sum b_n$ is a convergent p-series with $p=2>1$ and $0<a_n \le b_n$ for all n, $\sum a_n$ converges by the Direct Comparison Test

3. A 15 meter chain is hanging from the top of a building. If the chain has a density of 2 kg per meter, how much work in Joules is done in lifting the first 10 meters of the chain to the top of the building? Assume the acceleration due to gravity is g.

(A) $50g$ (B) $100g$ (C) $150g$ (D) $200g$ (E) $250g$



First 10 m: $W_i = F \cdot d = m \cdot a \cdot d = 2\Delta x \cdot g \cdot x_i$

$$\text{Work} = \int_0^{10} 2gx\,dx = gx^2\Big|_0^{10} = 100g$$

Last 5 m: $\text{Work} = F \cdot d = m \cdot a \cdot d = 5 \cdot 2 \cdot g \cdot 10 = 100g$

Total work $= 100g + 100g = 200g$

4. Which of the following statements is true concerning the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 3^n}{(2n+2)!}$?

(A) The Ratio Test Yields 0, so the series converges

(B) The Ratio Test Yields $\frac{1}{3}$, so the series converges

(C) The Ratio Test Yields $\frac{3}{4}$, so the series converges

(D) The Ratio Test Yields 1, so the test is inconclusive

(E) The Ratio Test Yields 3, so the series diverges

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2 \cdot 3^{n+1}}{(2n+4)!} \cdot \frac{(2n+2)!}{(n!)^2 \cdot 3^n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cdot (n!)^2 \cdot 3 \cdot 3^n}{(2n+4)(2n+3)(2n+2)!} \cdot \frac{(2n+2)!}{(n!)^2 \cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+4)(2n+3)} = \frac{3}{4} < 1 \Rightarrow \text{Series converges by the Ratio Test}$$

5. Let $\sum_{n=1}^{\infty} a_n$ be a series and let $S_N = 1 + \frac{1}{N}$ be the Nth partial sum. Which of the following statements must be true?

$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(1 + \frac{1}{N}\right) = 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n = 1$

$\Rightarrow \lim_{n \to \infty} a_n = 0$

(A) $\{a_n\}$ converges to 0 and $\sum_{n=1}^{\infty} a_n$ converges to 0

(B) $\{a_n\}$ converges to 0 and $\sum_{n=1}^{\infty} a_n$ converges to 1

(C) $\{a_n\}$ converges to 1 and $\sum_{n=1}^{\infty} a_n$ converges to 0

(D) $\{a_n\}$ converges to 1 and $\sum_{n=1}^{\infty} a_n$ converges to 1

(E) $\{a_n\}$ and $\sum_{n=1}^{\infty} a_n$ both diverge

6. To determine whether the series $\sum_{n=1}^{\infty} \frac{5n\sqrt{n+5}}{n^5+5n^2+1}$ converges or diverges using the **Limit Comparison Test**, we should compare $a_n = \frac{5n\sqrt{n+5}}{n^5+5n^2+1}$ with:

(A) $b_n = \frac{1}{n^{7/2}}$ (B) $b_n = \frac{1}{\sqrt{n}}$ (C) $b_n = \frac{1}{n^{9/2}}$ (D) $b_n = \frac{1}{n^5}$ (E) $b_n = \frac{1}{n}$

$a_n = \frac{5n\sqrt{n+5}}{n^5+5n^2+1} \sim \frac{5n\sqrt{n}}{n^5} = 5 \cdot \frac{n^{3/2}}{n^5} = \frac{5}{n^{5-3/2}} = \frac{5}{n^{7/2}} = 5b_n$

7. Which of the following series are absolutely convergent?

$$\mathbf{A}: \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}} \qquad \mathbf{B}: \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \qquad \mathbf{C}: \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

(A) **A** only (B) **B** only (C) **A** and **C** only (D) **B** and **C** only (E) **A**, **B**, and **C**

A: $\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ ⇒ converges by AST ; $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series Test ⇒ conditionally convergent

B: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges by AST ; $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by LCT ⇒ absolutely convergent

C: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely by Root Test

8. Let X be a continuous random variable with corresponding probability density function $f(x)$. How many of the following statements must be true?

✓ I. $f(x) \geq 0$ for all x

✓ II. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

✓ III. The probability X lies between 100 and 200 is given by $\int_{100}^{200} f(x)\,dx$

X IV. The median of X is given by $\int_{-\infty}^{\infty} x f(x)\,dx$. Median is m such that $\int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

9. Which of the following series converge?

$$\mathbf{A}: \sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^2} \qquad \mathbf{B}: \sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2} \qquad \mathbf{C}: \sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2}$$

(A) **A** and **B** only (B) **A** and **C** only (C) **B** and **C** only (D) **A**, **B**, and **C**

$\sum_{n=2}^{\infty} \frac{1}{n^p(\ln n)^q}$ converges if $p>1$ OR $p=1$ and $q>1$

A: $\sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^2} \Rightarrow p=-2, q=2 \Rightarrow$ diverges

B: $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2} \Rightarrow p=2, q=-2 \Rightarrow$ converges

C: $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)^2} \Rightarrow p=2, q=2 \Rightarrow$ converges

10. Which of the following statements is true concerning the series $\sum_{n=1}^{\infty} \frac{(n^2+3)3^{2n}}{(n^3+2)2^{3n}}$?

(A) The Root Test Yields 0, so the series converges

(B) The Root Test Yields $\frac{2}{3}$, so the series converges

(C) The Root Test Yields 1, so the test is inconclusive

(D) The Root Test Yields $\frac{9}{8}$, so the series diverges

(E) The Root Test Yields ∞, so the series diverges

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n^2+3)\,3^{2n}}{(n^3+2)\,2^{3n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2+3}\cdot 3^2}{\sqrt[n]{n^3+2}\cdot 2^3} = \frac{9}{8} > 1 \Rightarrow$$ Series diverges by Root Test

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Part II: Free Response There are **4** questions on this portion of the exam. Show **ALL** work clearly in the space provided for each problem, unless the problem says otherwise. Your work must be complete, logical, and understandable or it will receive no credit. Please cross out or fully erase any work that you do not want graded. A total of **55** points are available on this portion of the exam.

FR Scores	
1	/15
2	/15
3	/10
4	/15
FR Total	/55

1. Let X be a continuous random variable with a probability density function given by $f(x) = \dfrac{C}{(x+1)^3}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a). Find the value of C required for $f(x)$ to be a probability density function.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{C}{(x+1)^3}\,dx = \lim_{t\to\infty} \int_0^t C(x+1)^{-3}\,dx$$

$$= \lim_{t\to\infty} \frac{C(x+1)^{-2}}{-2} \Big|_0^t$$

$$= \lim_{t\to\infty} \left[-\frac{C}{2(t+1)^2} + \frac{C}{2} \right]$$

$$= 0 + \frac{C}{2} = 1 \Rightarrow \boxed{C = 2}$$

(b). Calculate $P(1 < X < 2)$, the probability that X lies between 1 and 2.

$$P(1 < x < 2) = \int_1^2 \frac{2}{(x+1)^3}\,dx = \int_1^2 2(x+1)^{-3}\,dx$$

$$= -(x+1)^{-2} \Big|_1^2$$

$$= -\frac{1}{3^2} + \frac{1}{2^2}$$

$$= \frac{1}{4} - \frac{1}{9} = \frac{9-4}{36} = \boxed{5/36}$$

2(a). Use the Alternating Series Test to show the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges. Be sure to state the conditions that need to be satisfied in order to use the Alternating Series Test. You do not need to check that these conditions are satisfied.

Let $b_n = \frac{1}{\sqrt{n}\ln n}$

1. $b_n \geq 0$
2. b_n is decreasing
3. $\lim_{n\to\infty} b_n = 0$

Therefore $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges by the Alternating Series Test

(b). Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converge absolutely or conditionally? Justify your answer.

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ diverges by the Generalized p-Series Test with $p = 1/2 < 1$

Because $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges but $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}\ln n} \right|$ diverges,

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges conditionally

(c). What is the maximum possible error of using the first 3 terms to approximate the sum of the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$?

Maximum possible error of using the first 3 terms is |fourth term| $= |a_5| = \frac{1}{\sqrt{5}\cdot\ln 5}$

3. Determine whether the following series converge or diverge. Be sure to justify your answer. If the series converges, then find the sum.

(a). $\frac{3}{5} - \frac{9}{25} + \frac{27}{125} - \frac{81}{625} + \ldots$

$$\frac{3}{5} - \frac{9}{25} + \frac{27}{125} - \frac{81}{625} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{5^n}$$

$$= \sum_{n=1}^{\infty} (-1) \cdot \left(-\frac{3}{5}\right)^n$$

converges by Geometric Series Test because $|r| = 3/5 < 1$

$$= \frac{\left(\frac{3}{5}\right)^1}{1-(-3/5)} = \frac{3/5}{8/5} = \boxed{\frac{3}{8}}$$

(b). $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n(n+2)}$

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

cover up: $A = \frac{2}{0+2} = 1$

$B = \frac{2}{-2} = -1$

$$\sum_{n=2}^{\infty} \frac{2}{n(n+2)} = \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_N = \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \ldots + \left(\cancel{\frac{1}{N-2}} - \cancel{\frac{1}{N}}\right)$$

$$+ \left(\cancel{\frac{1}{N-1}} - \frac{1}{N+1}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+2}\right)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$\sum_{n=2}^{\infty} \frac{2}{n(n+2)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2}\right)$$

$$= \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

4(a). Use the Direct Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{3^n}{8^n + n}$ converges or diverges. Justify your answer.

Let $a_n = \frac{3^n}{8^n+n} \geq 0$. Choose $b_n = \frac{3^n}{8^n} = \left(\frac{3}{8}\right)^n$. Then $\sum_{n=1}^{\infty} b_n$ is a convergent geometric series with $|r| = 3/8 < 1$

$a_n = \frac{3^n}{8^n+n} < \frac{3^n}{8^n} = b_n$

Because $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ for all n, $\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test.

(b). Use the Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges or diverges. Justify your answer.

Let $a_n = \sin(1/n^2) \geq 0$. Choose $b_n = 1/n^2$. Then $\sum_{n=1}^{\infty} b_n$ is a convergent p-series with $p=2>1$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1$. Because $\sum_{n=1}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.