Name: Solutions **Section #:** ____________

UF-ID: ______________________ **TA Name:** ____________

A: Sign the back of your scantron sheet.

B: On the indicated spaces on the front of your scantron, **in pencil**, write and encode:

1. Your name (last name, first initial, middle initial)
2. Your UFID Number
3. Your 4-digit Section Number

C: Under "special codes", code in the test ID number 4,1:

1 2 3 • 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, for "Test Form Code", encode A .

• B C D E

E: Some basic information about the exam:

1. This exam has 22 multiple choice worth 5 points each. The entire exam is out of 100 points.
2. You will have 120 minutes to take the exam.
3. You may write on your exam.
4. Raise your hand if you need more scratch paper or if you have a problem with the test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE EXAM.

F: KEEP YOUR SCANTRON COVERED AT ALL TIMES

G: When you are finished:

1. Before turning in your test, check for transcribing errors. Any mistakes you leave in are there to stay.
2. Turn in your scantron and the free-response portion of your exam to your TA, or the proctor designated as your TA. Be prepared to show your UF-ID card.
3. Solutions to the exam will be posted on Canvas after the exam is over.

The Honor Pledge: "On my honor, I have neither given nor received unauthorized aid doing this exam."

Signature: ______________________

There are **22** questions on this exam. Fill in the answers to these questions on the provided scantron sheet. Only answers on the scantron will be graded. Each problem is worth 5 points for a total of 110 points on the exam, but you grade will be taken out of 100 points.

1. Evaluate $\int_1^{e^2} x \ln x \, dx$.

(A) $\frac{e^4}{4} - \frac{1}{4}$ (B) $\frac{e^4}{4} + \frac{1}{4}$ (C) $\frac{3e^4}{4} - \frac{1}{4}$ (D) $\frac{3e^4}{4} + \frac{1}{4}$

$\int_1^e x \ln x \, dx$ $u = \ln x$ $dv = x\,dx$

$du = \frac{1}{x}\,dx$ $v = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln x \Big|_1^{e^2} - \int_1^{e^2} \frac{x}{2}\,dx$

$= \frac{e^4 \cdot \ln e^2}{2} - \frac{\ln 1}{2} - \frac{x^2}{4}\Big|_1^{e^2} = e^4 - \left[\frac{e^4}{4} - \frac{1}{4}\right] = \frac{3e^4}{4} + \frac{1}{4}$

2. Evaluate $\int \sec^7 x \tan^3 x \, dx$.

(A) $\frac{\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$ (B) $\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C$

(C) $\frac{\sec^9 x}{9} - \frac{\sec^7 x}{7} + C$ (D) $\frac{\tan^9 x}{9} - \frac{\tan^7 x}{7} + C$

$\int \sec^7 x \tan^3 x \, dx = \int \sec^6 x \tan^2 x \sec x \tan x \, dx$

$= \int \sec^6 x (\sec^2 x - 1) \sec x \tan x \, dx$ $u = \sec x$

$du = \sec x \tan x$

$= \int u^6 (u^2 - 1)\, du$

$= \int (u^8 - u^6)\, du$

$= \frac{u^9}{9} - \frac{u^7}{7} + C = \frac{\sec^9 x}{9} - \frac{\sec^7 x}{7} + C$

3. Which of the following trigonometric substitutions should be used to integrate $\int \frac{1}{\sqrt{x^2 - 8x - 9}}\, dx$?

(A) $x - 4 = 5\sec\theta$ (B) $x - 8 = 3\sec\theta$ (C) $x - 4 = 5\sin\theta$ (D) $x - 8 = 3\sin\theta$

$$x^2 - 8x - 9 = x^2 - 8x + 16 - 16 - 9 = (x-4)^2 - 25$$

$$\Rightarrow x - 4 = 5\sec\theta$$

4. Evaluate $\int_0^{\infty} \frac{e^x}{e^{2x}+1}\, dx$.

(A) $\frac{1}{2}$ (B) $\frac{\pi}{4}$ (C) 1 (D) $\frac{\pi}{2}$ (E) π

$$\lim_{t\to\infty} \int_0^t \frac{e^x}{e^{2x}+1}\, dx$$

$u = e^x \Rightarrow du = e^x dx$

$x = 0 \Rightarrow u = e^0 = 1$

$x = t \Rightarrow u = e^t$

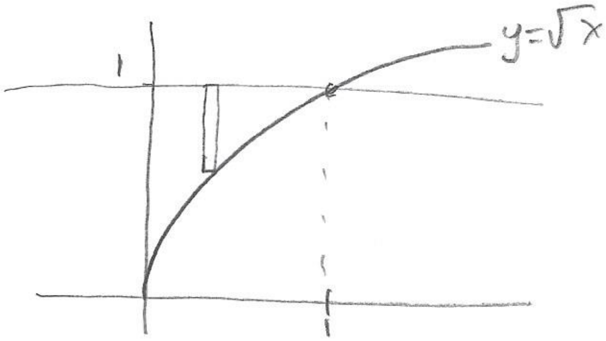
$$\lim_{t\to\infty} \int_1^{t^2} \frac{1}{u^2+1}\, du = \lim_{t\to\infty} \arctan u \Big|_1^{e^t}$$

$$= \lim_{t\to\infty} \left[\arctan(e^t) - \arctan 1\right]$$

$$= \pi/2 - \pi/4 = \pi/4$$

5. Let R be the region in the first quadrant bounded by the curves $y = \sqrt{x}$, $y = 1$, and the y-axis. What is the volume of the solid formed by rotating R about the x-axis?

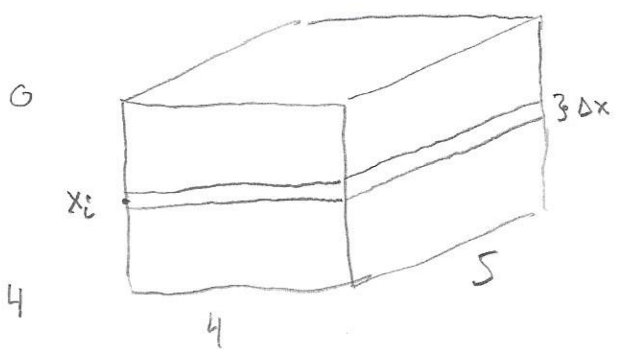
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) π (E) $\frac{3\pi}{2}$



$$\text{Volume} = \pi \int_0^1 \left[1^2 - \sqrt{x}^2\right] dx = \pi \int_0^1 (1-x)\, dx = \pi\left[x - x^2/2\right]_0^1$$
$$= \pi/2$$

6. A rectangular tank with a base 4 meters by 5 meters is completely filled with water. If the tank is 4 meterst tall, then how much work is required to pump half of the water out over the top of the tank? Assume the denisty of water is $\rho = 1$ and the acceleration due to gravity is g.

(A) $40g$ (B) $80g$ (C) $120g$ (D) $160g$ (E) $200g$



$$W_i = F_i \cdot d_i$$
$$= V_i \cdot \rho \cdot g \cdot x_i$$
$$= 4 \cdot 5 \cdot \Delta x \cdot g \cdot x_i$$
$$= 20g\, x_i\, \Delta x$$

$$\text{Work} = \int_0^2 20g\, x\, dx = 10g x^2 \Big|_0^2 = 40g$$

7. Consider two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Which of the following statemnets must be true?

I. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges X consider $a_n = 1/n$

II. If $0 < a_n < b_n$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges X consider $a_n = 1/n^2+1$, $b_n = 1$

III. If $0 < a_n < b_n$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges ✓ True by DCT

(A) II only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

8. Find the sum $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$.

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) The series diverges

$$\frac{1}{n^3-n} = \frac{1}{n(n^2-1)} = \frac{1}{n(n-1)(n+1)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+1}$$

Cover up: $A = -1$, $B = \frac{1}{2}$, $C = 1/2$ $\Rightarrow \frac{1}{n^3-n} = -\frac{1}{n} + \frac{1}{2}\cdot\frac{1}{n-1} + \frac{1}{2}\cdot\frac{1}{n+1}$

$$\sum_{n=2}^{\infty} \frac{1}{n^3-n} = \sum_{n=2}^{\infty} \left[\frac{1}{2}\cdot\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n}\right)\right]$$

$$= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\left[\left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \dots + \left(\cancel{\tfrac{1}{N-1}} - \frac{1}{N}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{2}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{3}}\right) + \dots + \left(\frac{1}{N+1} - \cancel{\tfrac{1}{N}}\right)\right]$$

$$= \frac{1}{2}\lim_{N\to\infty}\left[1 - \frac{1}{N} - \frac{1}{2} + \frac{1}{N+1}\right] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

9. For which values of k does that series $\displaystyle\sum_{n=5}^{\infty} \frac{n^{1/2}}{\sqrt[3]{n^k + 7n}}$ converge?

(A) $k > \frac{9}{2}$ (B) $k \geq \frac{9}{2}$ (C) $k > 5$ (D) $k \geq 5$ (E) $k > 6$

$$\frac{n^{1/2}}{\sqrt[3]{n^k+7n}} \approx \frac{n^{1/2}}{\sqrt[3]{n^k}} = \frac{n^{1/2}}{n^{k/3}} = \frac{1}{n^{k/3-1/2}}$$

Series converges when $\frac{k}{3} - \frac{1}{2} > 1$

$$\frac{k}{3} > \frac{3}{2}$$

$$k > 9/2$$

10. Which of the following sequences converge?

✓ I. $\{a_n\} = \left\{\frac{1}{n}\right\}$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$

✓ II. $\{b_n\} = \left\{\frac{2n}{n+1}\right\}$ $\quad \lim_{n\to\infty} \frac{2n}{n+1} = 2$

✓ III. $\{c_n\} = \left\{\frac{1}{2^n}\right\}$ $\quad \lim_{n\to\infty} \frac{1}{2^n} = 0$

(A) $\{c_n\}$ only (B) $\{a_n\}$ and $\{b_n\}$ only (C) $\{a_n\}$ and $\{c_n\}$ only (D) $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$

11. Which of the following statements is true concerning the three series below?

$$\mathbf{A}: \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5} \qquad \mathbf{B}: \sum_{n=1}^{\infty} (-1)^n \cos(1/n^4) \qquad \mathbf{C}: \sum_{n=1}^{\infty} \frac{7+(-1)^n}{n^2}$$

(*A*) **A** converges absolutely, **B** converges conditionally, and **C** diverges
(*B*) **A** converges conditionally, **B** converges absolutely, and **C** diverges
(*C*) **A** diverges, **B** converges absolutely, and **C** converges conditionally
(*D*) **A** converges absolutely, **B** diverges, and **C** converges conditionally
(*E*) **A** converges conditionally, **B** diverges, and **C** converges absolutely

A: Converges by AST, $\sum_{n=1}^{\infty} \frac{n}{n^2+5}$ diverges by LCT ($b_n = 1/n$) $\Rightarrow$ converges conditionally

B: $\lim_{n\to\infty} (-1)^n \cdot \cos(1/n^4) = \lim_{n\to\infty} (-1)^n \cdot \cos 0 = \lim_{n\to\infty} (-1)^n$ diverges due to oscillation
$\Rightarrow$ diverges by TFD

C: Converges by DCT ($b_n = 8/n^2$), Series is positive $\Rightarrow$ $\sum a_n = \sum |a_n|$
$\Rightarrow$ Converges absolutely

12. Which of the following statements is true concerning the series $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^n (n+2)^4}{8^n}$?

(*A*) The Root Test yields 0, so the series converges
(*B*) The Root Test yields $\frac{1}{8}$, so the series converges
(*C*) The Root Test yields 1, so the test is inconclusive
(*D*) The Root Test yields 8, so the series diverges
(*E*) The Root Test yields ∞, so the series diverges

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n (n+2)^4}{8^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{(n+2)^4}}{8} = \frac{1}{8} < 1 \Rightarrow$$ Converges by Root Test

13. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)x^n}{3^n n!}$?

(A) 0 (B) $\frac{2}{3}$ (C) 1 (D) $\frac{3}{2}$ (E) ∞

$$\lim_{n\to\infty} \left| \frac{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}(2n+3)\, x^{\cancel{n+1}}}{3^{\cancel{n+1}}\cdot \cancel{(n+1)!}} \cdot \frac{\cancel{3^n \cdot n!}}{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}\, \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} |x| \cdot \frac{(2n+3)}{3(n+1)} = \frac{2}{3}|x| < 1 \Rightarrow |x| < 3/2 \leftarrow \text{ROC}$$

14. Let $f(x) = x^3 \sin(x^2)$. Use the Alternating Series Estimation Theorem to find the maximum possible error of using the first two nonzero terms of the Maclaurin series of $f(x)$ to approximate $f(0.1)$.

(A) $\frac{1}{120}$ (B) $\frac{1}{120 \cdot 10^5}$ (C) $\frac{1}{120 \cdot 10^9}$ (D) $\frac{1}{120 \cdot 10^{13}}$ (E) $\frac{1}{120 \cdot 10^{17}}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^3 \sin(x^2) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+5}}{(2n+1)!}$$

$$f(0.1) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 0.1^{4n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 10^{4n+5}} = \frac{1}{10^5} - \frac{1}{3! \cdot 10^9} + \boxed{\frac{1}{5! \cdot 10^{13}}} - \cdots$$

maximum possible error

15. Let $f(x) = \sqrt{x}$. Which of the following is the Taylor series centered at 1?

(A) $1 + \frac{(x-1)}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}(x-1)^n$

(B) $1 + \frac{(x-1)}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}(x-1)^n$

(C) $1 + \frac{(x-1)}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}(x-1)^n$

(D) $1 + \frac{(x-1)}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}(x-1)^n$

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2}$

$f'''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} x^{-5/2}$

$f^{(4)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} x^{-7/2}$

$f^{(n)}(x) = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} \cdot x^{\frac{-(2n-1)}{2}}$

$f(1) = 1$

$f'(1) = 1/2$

$f^{(n)}(1) = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}, \quad n \geq 2$

$\Rightarrow \sqrt{x} = 1 + \frac{1}{2}(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-1)^n}{2^n \cdot n!}$

16. Let $f(x) = 1 + 8(x+3)^7 - 18(x+3)^{18} + 2(x+3)^{31}$. How many of the following statements must be true?

I. $f'(-3) = 0$

II. $f^{(7)}(-3) = 8!$

III. $f^{(18)}(-3) = -18 \cdot 18!$

IV. $f^{(31)}(-3) = 2 \cdot 31!$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

$a_n = \frac{f^{(n)}(-3)}{n!} \Rightarrow f^{(n)}(-3) = a_n \cdot n!$

$f'(-3) = a_1 \cdot 1! = 0$

$f^{(7)}(-3) = a_7 \cdot 7! = 8 \cdot 7! = 8!$

$f^{(18)}(-3) = a_{18} \cdot 18! = -18 \cdot 18!$

$f^{(31)}(-3) = a_{31} \cdot 31! = 2 \cdot 31!$

17. Which of the following lines is tangent to the parametric curve $x(t) = te^t$, $y(t) = e^{t^3}$ at the point (e, e)?

(A) $x(t) = t$, $y(t) = \frac{3}{2}t + e$

(B) $x(t) = t + e$, $y(t) = \frac{3}{2}t + e$

(C) $x(t) = t + e$, $y(t) = \frac{2}{3}t$

(D) $x(t) = t + e$, $y(t) = \frac{2}{3}t + e$

(E) $x(t) = t - e$, $y(t) = t - e$

(e,e) occurs when $t=1$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 e^{t^3}}{te^t + e^t}$$

when $t=1$, $\frac{dy}{dx} = \frac{3e}{e+e} = \frac{3}{2}$

Tangent line: $x(t) = t+e$, $y(t) = \frac{3}{2}t + e$

18. Convert the Cartesian equation $(x^2 + y^2)^3 = 4x^2y^2$ to a polar equation.

(A) $r = 2\sin\theta$ (B) $r = 2\cos\theta$ (C) $r = \sin(2\theta)$ (D) $r = \cos(2\theta)$

$(x^2+y^2)^3 = 4x^2y^2$

$(r^2)^3 = 4\cdot(r\cos\theta)^2(r\sin\theta)^2$

$r^6 = 4r^4\cos^2\theta\sin^2\theta$

$r^2 = (2\cos\theta\sin\theta)\cdot(2\cos\theta\sin\theta)$

$r^2 = \sin^2(2\theta) \Rightarrow r = \sin(2\theta)$

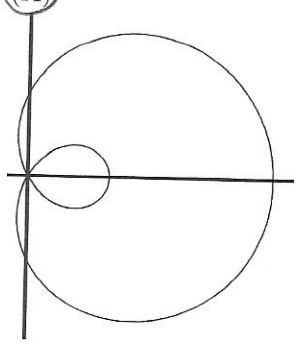
$x^2+y^2 = r^2$

$x = r\cos\theta$

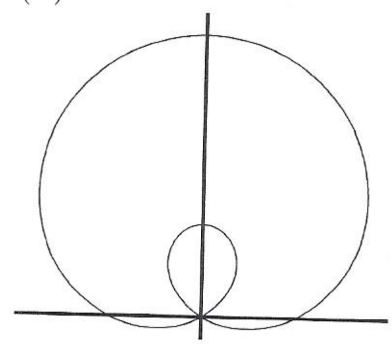
$y = r\sin\theta$

19. Which of the following is the plot of the polar curve $r = 2\cos\theta - 1$?

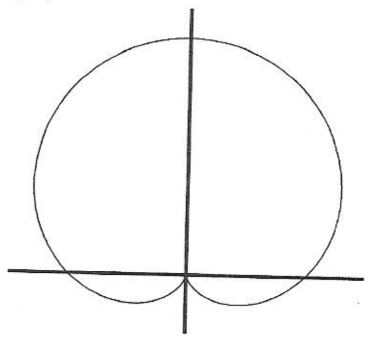
(A)



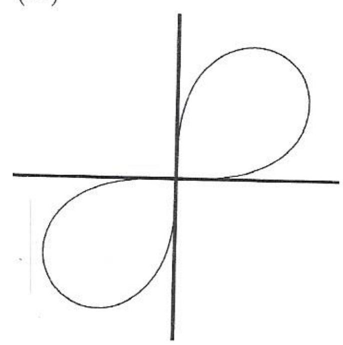
(B)



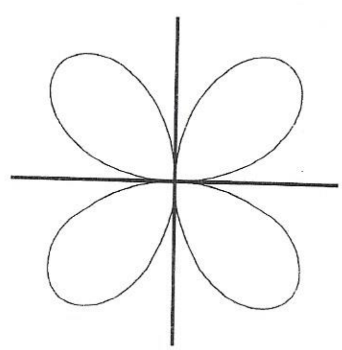
(C)



(D)



20. The graph of which of the following polar equations is shown below?



$\theta = 0 \Rightarrow r = 0$

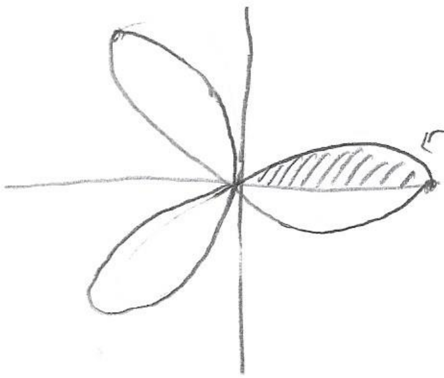
$\Rightarrow$ sine function

4 petals $\Rightarrow n = 2$

(A) $r = \sin(2\theta)$ (B) $r = \cos(2\theta)$ (C) $r = \sin(4\theta)$ (D) $r = \cos(4\theta)$

21. Which of the following integrals gives the area of one petal of the polar curve $r = \cos(3\theta)$?

(A) $\int_0^{\pi/6} \cos(3\theta)\, d\theta$ (B) $\int_0^{\pi/6} \cos^2(3\theta)\, d\theta$ (C) $\int_0^{2\pi} \cos(3\theta)\, d\theta$ (D) $\int_0^{2\pi} \cos^2(3\theta)\, d\theta$



traced when $0 \le \theta \le \pi/6$

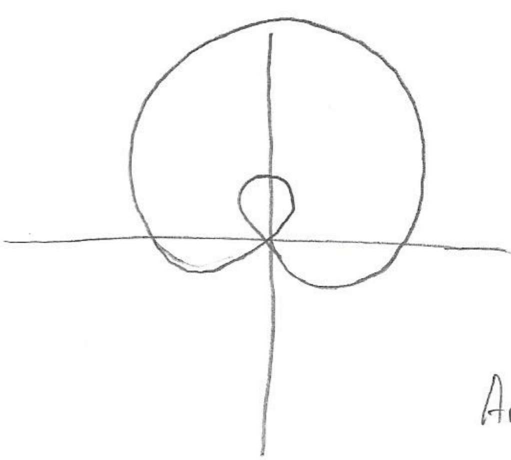
Area of shaded region $= \frac{1}{2}\int_0^{\pi/6} [\cos(3\theta)]^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi/6} \cos^2(3\theta)\, d\theta$

Area of one petal $= 2\cdot\frac{1}{2}\int_0^{\pi/6} \cos^2(3\theta)\, d\theta = \int_0^{\pi/6} \cos^2(3\theta)\, d\theta$

22. The area of the inner loop of the polar curve $r = 1 + 2\sin\theta$ is given by $\frac{1}{2}\int_{7\pi/6}^{\beta} (1 + 2\sin\theta)^2\, d\theta$ for which of the following values of β?

(A) $\frac{4\pi}{3}$ (B) $\frac{3\pi}{2}$ (C) $\frac{5\pi}{3}$ (D) $\frac{11\pi}{6}$ (E) 2π



$1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

inner loop is traced when $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$

Area $= \frac{1}{2}\int_{7\pi/6}^{\boxed{11\pi/6}} (1 + 2\sin\theta)^2\, d\theta$

$f(x)$	Maclaurin Series	Interval of Convergence
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \dfrac{x^n}{n}$	$[-1,1)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$	$(-1,1]$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$(-\infty,\infty)$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$[-1,1]$

Taylor Remainder Estimate:

If $\left|f^{(N+1)}(a)\right| \leq M$ for all numbers a between x and c, then the remainder $R_N(x)$ of the Taylor series satisfies the inequality $|R_N(x)| \leq \dfrac{M|x-c|^{N+1}}{(N+1)!}$ for all numbers between x and c.

Taylor Coefficients:

$$a_n = \frac{f^{(n)}(c)}{n!}$$