Name: Solutions **Section #:** ____________

UF-ID: ____________________ **TA Name:** ____________

A: Sign the back of your scantron sheet.

B: On the indicated spaces on the front of your scantron, **in pencil**, write and encode:

1. Your name (last name, first initial, middle initial)
2. Your UFID Number
3. Your 4-digit Section Number

C: Under "special codes", code in the test ID number 3,1:

1	2	•	4	5	6	7	8	9	0
•	2	3	4	5	6	7	8	9	0

D. At the top right of your scantron, for "Test Form Code", encode A .

• B C D E

E: Some basic information about the exam:

1. This exam has two parts: a 10-question multiple choice section worth 50 points, and a 4-question free response section worth 55 points. The entire exam is out of 100 points.

2. You will have 100 minutes to take the exam.

3. You may write on your exam.

4. Raise your hand if you need more scratch paper or if you have a problem with the test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE EXAM.

F: KEEP YOUR SCANTRON COVERED AT ALL TIMES

G: When you are finished:

1. Before turning in your test, check for transcribing errors. Any mistakes you leave in are there to stay.

2. Turn in your scantron and the free-response portion of your exam to your TA, or the proctor designated as your TA. Be prepared to show your UF-ID card.

3. Solutions to the exam will be posted on Canvas after the exam is over.

The Honor Pledge: "On my honor, I have neither given nor received unauthorized aid doing this exam."

Signature: ______________________________

Part I: Multiple Choice There are **10** questions on this portion of the exam. Fill in the answers to these questions on the provided scantron sheet. Only answers on the scantron will be graded. Each problem is worth 5 points, for a total of 50 points on this portion of the exam.

1. Find the sum of the series below.

$$4 - \frac{\pi^2}{2! \cdot 4} + \frac{\pi^4}{4! \cdot 4^3} - \frac{\pi^6}{6! \cdot 4^5} + \cdots$$

(A) $\frac{\sqrt{2}}{2}$ (B) $\sqrt{2}$ (C) $2\sqrt{2}$ (D) $4\sqrt{2}$ (E) $6\sqrt{2}$

$$4 - \frac{\pi^2}{2! \cdot 4} + \frac{\pi^4}{4! \cdot 4^3} - \frac{\pi^6}{6! \cdot 4^5} + \cdots = \sum_{n=0}^{\infty} \frac{4 \cdot \pi^{2n}}{(2n)!\, 4^{(2n)}} = 4 \sum_{n=0}^{\infty} \frac{(\pi/4)^{2n}}{(2n)!}$$

Because $\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $4 \sum_{n=0}^{\infty} \frac{(\pi/4)^{2n}}{(2n)!} = 4 \cdot \cos(\pi/4)$

$= 4 \cdot \frac{\sqrt{2}}{2}$

$= 2\sqrt{2}$

2. The parametric curve $x(t) = \sin t$, $y(t) = \cos^2 t$ is part of what type of curve?

(A) Circle (B) Ellipse (C) Line (D) Parabola

$X = \sin t \Rightarrow X^2 = \sin^2 t$

$\sin^2 t + \cos^2 t = 1 \Rightarrow x^2 + y = 1$

$\Rightarrow y = 1 - x^2$ ← parabola

3. Use the first two nonzero terms of the Maclaurin series for $f(x) = \dfrac{\ln(1+x^2)}{x^2}$ to approximate $f(0.2)$.

(A) 0.95 (B) 0.96 (C) 0.97 (D) 0.98 (E) 0.99

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n}$$

Therefore $$\frac{\ln(1+x^2)}{x^2} = \frac{1}{x^2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{n} = 1 - \frac{x^2}{2} + \dots$$

$$f(0.2) \approx 1 - \frac{0.2^2}{2} = 1 - \frac{0.04}{2} = 1 - 0.02 = 0.98$$

4. Which of the following is a power series representation for $f(x) = \dfrac{1}{(6+x)^2}$?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{n-1}}{6^{n+1}}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n}}{6^{n+1}}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n+1}}{(n+1)6^{n+1}}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{6^{2n+2}}$

(E) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{n-1}}{6}$

$$\frac{1}{6+x} = \frac{1}{6(1+x/6)} = \frac{1}{6} \cdot \frac{1}{1-(-x/6)} = \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{-x}{6}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{6^{n+1}}$$

$$\frac{1}{(6+x)^2} = -\frac{d}{dx}\left[\frac{1}{6+x}\right] = -\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{6^{n+1}}\right] = -\sum_{n=1}^{\infty} \frac{(-1)^n n \cdot x^{n-1}}{6^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \cdot x^{n-1}}{6^{n+1}}$$

5. Which of the following lines is tangent to the parametric curve $x(t) = te^t$, $y(t) = e^{t^3}$ at the point (e, e)?

(A) $x(t) = t$, $y(t) = \frac{3}{2}t + e$

(B) $x(t) = t + e$, $y(t) = \frac{3}{2}t + e$

(C) $x(t) = t + e$, $y(t) = \frac{2}{3}t$

(D) $x(t) = t + e$, $y(t) = \frac{2}{3}t + e$

(E) $x(t) = t - e$, $y(t) = t - e$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{t^3} \cdot 3t^2}{te^t + e^t}$$

The point (e,e) corresponds to $t=1$.

when $t=1$, $\frac{dy}{dx} = \frac{e^1 \cdot 3}{e^1 + e^1} = \frac{3}{2}$

Tangent line: $X(t) = e + t$

$y(t) = e + \frac{3}{2}t$

6. What is the maximum possible error of using the 5th degree Taylor polynomial for $f(x) = \cos x$ centered at $\frac{\pi}{3}$ to approximate $\cos 1$? Use the fact that $|\sin x| \le 1$ and $|\cos x| \le 1$ for all real numbers x.

(A) $\frac{\left|\frac{1}{2} - \frac{\pi}{3}\right|^5}{5!}$ (B) $\frac{\left|1 - \frac{\pi}{3}\right|^5}{5!}$ (C) $\frac{\left|1 - \frac{1}{2}\right|^5}{5!}$ (D) $\frac{\left|\frac{1}{2} - \frac{\pi}{3}\right|^6}{6!}$ (E) $\frac{\left|1 - \frac{\pi}{3}\right|^6}{6!}$

$$|R_{N+1}| \le \frac{M \cdot |x-c|^{N+1}}{(N+1)!}$$ where $|f^{(N+1)}(a)| \le M$ for all a between x and c

$N=5 \Rightarrow |f^{(6)}(x)| = |-\cos x| \le 1 = M$ for all real x

$$|R_6| \le \frac{1 \cdot |1 - \pi/3|^6}{6!}$$

7. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{3^n\sqrt{n}}$.

(A) $(-3,3)$ (B) $(-1,5)$ (C) $(-1,5]$ (D) $[-1,5)$ (E) $[-1,5]$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{3^{n+1}\sqrt{n+1}}\cdot\frac{3^n\sqrt{n}}{(-1)^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\frac{|x-2|}{3}\cdot\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x-2|}{3} < 1$$

$\Rightarrow |x-2| < 3$

$\Rightarrow -1 < x < 5$

when $x=5$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 3^n}{3^n\sqrt{n}}$ converges by AST

$x=-1$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-3)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} -\frac{1}{\sqrt{n}}$ diverges by P-series Test $\Rightarrow$ IOC: $(-1,5]$

8. Which of the following is the Taylor series for $f(x)=\frac{1}{x}$ centered at 5?

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-5)^n}{5^n}$

(B) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-5)^n}{5^{n+1}}$

(C) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-5)^n}{n!5^n}$

(D) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-5)^n}{n!5^{n+1}}$

(E) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-5)^n}{(n+1)!5^{n+1}}$

	$f^{(n)}(x)$	$f^{(n)}(5)$	a_n
$n=0$	$\frac{1}{x}$	$\frac{1}{5}$	$\frac{1}{5\cdot 0!}$
$n=1$	$-\frac{1}{x^2}$	$-\frac{1}{5^2}$	$-\frac{1}{5^2\cdot 1!}$
$n=2$	$\frac{1\cdot 2}{x^3}$	$\frac{1\cdot 2}{5^3}$	$\frac{1\cdot 2}{5^3\cdot 2!}$
$n=3$	$-\frac{1\cdot 2\cdot 3}{x^4}$	$-\frac{1\cdot 2\cdot 3}{5^4}$	$-\frac{1\cdot 2\cdot 3}{5^4\cdot 3!}$
	$\vdots$		

$f^{(n)}(x) = \frac{(-1)^n\cdot n!}{x^{n+1}}$ $f^{(n)}(5) = \frac{(-1)^n n!}{5^{n+1}}$ $a_n = \frac{(-1)^n\cdot \cancel{n!}}{5^{n+1}\cdot\cancel{n!}}$

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}\cdot(x-5)^n$$

9. Which of the following polar points does **NOT** represent the Cartesian point $(x, y) = (1, \sqrt{3})$?

(A) $(r, \theta) = (-2, 4\pi/3)$ (B) $(r, \theta) = (2, -\pi/3)$ (C) $(r, \theta) = (2, \pi/3)$ (D) $(r, \theta) = (2, 7\pi/3)$

A: $x = r\cos\theta = -2\cdot\cos 4\pi/3 = -2\cdot -\frac{1}{2} = 1$ $y = r\sin\theta = -2\cdot\sin 4\pi/3 = -2\cdot -\frac{\sqrt{3}}{2} = \sqrt{3}$

B: $x = r\cos\theta = 2\cdot\cos(-\pi/3) = 2\cdot\frac{1}{2} = 1$ $y = 2\cdot\sin(-\pi/3) = 2\cdot -\frac{\sqrt{3}}{2} = -\sqrt{3}$

C: $x = r\cos\theta = 2\cdot\cos\pi/3 = 2\cdot 1/2 = 1$ $y = r\sin\theta = 2\cdot\sin\pi/3 = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

D: $x = r\cos\theta = 2\cdot\cos 7\pi/3 = 2\cdot 1/2 = 1$ $y = r\cos\theta = 2\cdot\sin 7\pi/3 = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

10. Let $f(x) = e^{-x^2/3}$. Calculate $f^{(2022)}(0)$.

(A) $\frac{-2022!}{3^{1011}1011!}$ (B) $\frac{-1}{3^{1011}1011!}$ (C) 0 (D) $\frac{1}{3^{1011}1011!}$ (E) $\frac{2022!}{3^{2022}2022!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2/3} = \sum_{n=0}^{\infty} \frac{(-x^2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n \cdot n!}$$

$$a_{2022} = \frac{f^{(2022)}(0)}{2022!} \Rightarrow f^{(2022)}(0) = a_{2022} \cdot 2022!$$

a_{2022} is the coefficient of the x^{2022} term $\Rightarrow 2n = 2022 \Rightarrow n = 1011$

$$a_{2022} = \frac{(-1)^{1011}}{3^{1011} \cdot 1011!}$$

$$f^{(2022)}(0) = \frac{-1}{3^{1011} \cdot 1011!} \cdot 2022!$$

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Part II: Free Response There are **4** questions on this portion of the exam. Show **ALL** work clearly in the space provided for each problem, unless the problem says otherwise. Your work must be complete, logical, and understandable or it will receive no credit. Please cross out or fully erase any work that you do not want graded. A total of **55** points are available on this portion of the exam.

FR Scores	
1	/20
2	/15
3	/10
4	/10
FR Total	/55

1. (*a*) Find a power series representation for $\int 18\sqrt{x}\arctan\sqrt{x}\,dx$ and find its interval of convergence.

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow 18\sqrt{x}\arctan\sqrt{x} = 18\sqrt{x}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot(\sqrt{x})^{2n+1}}{2n+1}$$

$$= 18\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2n+1},$$

$$\int 18\sqrt{x}\arctan\sqrt{x}\,dx = \int 18\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2n+1}\,dx$$

converges when $-1 \le \sqrt{x} \le 1$
$\Rightarrow$ IOC: $[0,1]$

$$\boxed{18\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{(n+2)(2n+1)} + C}$$

Integrating a power series keeps the radius of convergence the same but could introduce convergence at endpoints. Because original IOC is a closed interval, $\boxed{IOC = [0,1]}$

(*b*) Use the first 2 nonzero terms of the power series from part (*a*) to approximate $\int_0^{0.1} 18\sqrt{x}\arctan\sqrt{x}\,dx$.

$$\int 18\sqrt{x}\arctan\sqrt{x} = 18\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{(n+2)(2n+1)} + C \approx 18\cdot\frac{x^2}{2} - 18\frac{x^3}{9} + C$$

$$= 9x^2 - 2x^3 + C$$

Therefore $\int_0^{0.1} 18\sqrt{x}\arctan\sqrt{x}\,dx \approx \left[9x^2 - 2x^3\right]_0^{0.1}$

$$= 9\cdot 0.1^2 - 2\cdot 0.1^3$$

$$= 0.09 - 0.002 = \boxed{0.088}$$

(*c*) Find the maximum possible error of your answer from part (*b*).

Next nonzero term of the series is $18\cdot\frac{x^4}{20}\Big|_0^{0.1}$

So by the Alternating Series Estimation Theorem

maximum error $= 18\cdot\frac{0.1^4}{20}$

2. (a) Find the second degree Taylor polynomial of $\arctan x$ centered at 1.

Let $f(x) = \arctan x$ $\qquad f(1) = \arctan 1 = \frac{\pi}{4}$ $\qquad a_0 = \frac{\pi/4}{0!} = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(1) = \frac{1}{2}$ $\qquad a_1 = \frac{1/2}{1!} = \frac{1}{2}$

$f''(x) = -\frac{1}{(1+x^2)^2} \cdot 2x$ $\qquad f''(1) = -\frac{2}{2^2} = \frac{-1}{2}$ $\qquad a_2 = \frac{-1/2}{2!} = -\frac{1}{4}$

$$T_2(x) = \pi/4 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$$

(b) Use the second degree Taylor polynomial of $\arctan x$ centered at 1 to approximate $\arctan 1.2$. Use $\frac{\pi}{4} = 0.785$ in your approximation and express your answer as a decimal. You may assume that 1.2 is in the interval of convergence of the Taylor series.

$$\arctan 1.2 \approx T_2(1.2) = \pi/4 + \frac{1}{2}(1.2-1) - \frac{1}{4}(1.2-1)^2$$

$$= 0.785 + \frac{1}{2} \cdot 0.2 - \frac{1}{4} \cdot 0.04$$

$$= 0.785 + 0.1 - 0.01$$

$$= 0.875$$

3. Is the parametric curve $x(t) = \cos t$, $y(t) = t \sin t$ concave upward or concave downward when $t = \pi/2$?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t\cos t + \sin t}{-\sin t} = -t\cot t - 1$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}\left[-t\cot t - 1\right]}{-\sin t}$$

$$= \frac{t\csc^2 t - \cot t}{-\sin t}$$

When $t = \pi/2$,

$$\frac{d^2y}{dx^2} = \frac{\pi/2 \cdot \csc^2(\pi/2) - \cot(\pi/2)}{-\sin(\pi/2)}$$

$$= \frac{\pi/2 \cdot 1 - 0}{-1}$$

$$= -\pi/2 < 0$$

Because $\frac{d^2y}{dx^2} < 0$, the curve is concave downward when $t = \pi/2$.

4. Find the length of the parametric curve $x(t) = 2\cos^3 t$, $y(t) = 2\sin^3 t$ from $t = 0$ to $t = \frac{\pi}{2}$.

$$L = \int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{(6\cos^2 t \cdot -\sin t)^2 + (6\sin^2 t \cdot \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{36\cos^4 t \sin^2 t + 36 \sin^4 t \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{36\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}\, dt$$

$$= \int_0^{\pi/2} 6\cos t \sin t\, dt$$

$u = \sin t$

$du = \cos t\, dt$

when $t = 0$, $u = 0$

$t = \pi/2$, $u = 1$

$$= \int_0^1 6u\, du$$

$$= 3u^2 \Big|_0^1 = \boxed{3}$$

$f(x)$	Maclaurin Series	Interval of Convergence
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \dfrac{x^n}{n}$	$[-1,1)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$	$(-1,1]$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$(-\infty,\infty)$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$[-1,1]$

Taylor Remainder Estimate:

If $\left|f^{(N+1)}(a)\right| \leq M$ for all numbers a between x and c, then the remainder $R_N(x)$ of the Taylor series satisfies the inequality $|R_N(x)| \leq \dfrac{M|x-c|^{N+1}}{(N+1)!}$ for all numbers between x and c.

Taylor Coefficients:

$$a_n = \frac{f^{(n)}(c)}{n!}$$