Name: Solutions **Section #:** ____________

UF-ID: ____________ **TA Name:** ____________

A: Sign the back of your scantron sheet.

B: On the indicated spaces on the front of your scantron, **in pencil**, write and encode:

1. Your name (last name, first initial, middle initial)
2. Your UFID Number
3. Your 4-digit Section Number

C: Under "special codes", code in the test ID number 2,1:

1 • 3 4 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, for "Test Form Code", encode A .

• B C D E

E: Some basic information about the exam:

1. This exam has two parts: a 10-question multiple choice section worth 50 points, and a 4-question free response section worth 55 points. The entire exam is out of 100 points.

2. You will have 100 minutes to take the exam.

3. You may write on your exam.

4. Raise your hand if you need more scratch paper or if you have a problem with the test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE EXAM.

F: KEEP YOUR SCANTRON COVERED AT ALL TIMES

G: When you are finished:

1. Before turning in your test, check for transcribing errors. Any mistakes you leave in are there to stay.

2. Turn in your scantron and the free-response portion of your exam to your TA, or the proctor designated as your TA. Be prepared to show your UF-ID card.

3. Solutions to the exam will be posted on Canvas after the exam is over.

The Honor Pledge: "On my honor, I have neither given nor received unauthorized aid doing this exam."

Signature: ____________

Part I: Multiple Choice There are **10** questions on this portion of the exam. Fill in the answers to these questions on the provided scantron sheet. Only answers on the scantron will be graded. Each problem is worth 5 points, for a total of 50 points on this portion of the exam.

1. Find a formula for the nth term a_n of the sequence below assuming the pattern of the first few terms continues. Assume the first term is a_1.

$$\left\{\frac{1}{1}, -\frac{2}{4}, \frac{4}{9}, -\frac{8}{16}, \frac{16}{25}, -\frac{32}{36}, \dots\right\}$$

(A) $a_n = \dfrac{(-1)^n 2^{n-1}}{n^2}$ (B) $a_n = \dfrac{(-1)^{n+1} 2^{n-1}}{n^2}$ (C) $a_n = \dfrac{(-1)^n 2^n}{n^2}$ (D) $a_n = \dfrac{(-1)^{n+1} 2^n}{n^2}$

Numerator: 2^{n-1}

Denominator: n^2

Alternator: $(-1)^{n+1}$

$$\Rightarrow a_n = \frac{(-1)^{n+1} \cdot 2^{n-1}}{n^2}$$

2. Which of the following statements is true concerning the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$?

(A) The Root Test yields 0, so the series converges

(B) The Root Test yields $\frac{1}{e}$, so the series converges

(C) The Root Test yields 1, so the test is inconclusive

(D) The Root Test yields e, so the series diverges

(E) The Root Test yields ∞, so the series diverges

$$\lim_{n\to\infty} \left[\frac{n}{(\ln n)^n}\right]^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{\ln n} = 0 \quad \text{because} \lim_{n\to\infty} n^{1/n} = 1.$$

$\Rightarrow$ Series converges

3. Which of the following series diverges by the Limit Comparison Test when comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$?

(A) $\sum_{n=1}^{\infty} \frac{\tan(1/n)}{\sqrt{n}}$ (B) $\sum_{n=1}^{\infty} \frac{\tan(1/n)}{n}$ (C) $\sum_{n=1}^{\infty} n\sin(1/n^2)$ (D) $\sum_{n=1}^{\infty} \sin^2(1/n)$ (E) $\sum_{n=1}^{\infty} \cos(1/n)$

We need $\lim_{n\to\infty} \frac{a_n}{1/n}$ to be positive and finite:

$$\lim_{n\to\infty} \frac{n\sin(1/n^2)}{1/n} = \lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1$$

4. Which of the following statements is true concerning the series $\sum_{n=1}^{\infty} e^{1-n}$?

(A) The Integral Test yields $\frac{1}{e}$, so the series converges

(B) The Integral Test yields $\frac{1}{e}$, so the series diverges

(C) The Integral Test yields 1, so the series converges

(D) The Integral Test yields 1, so the test is inconclusive

(E) The Integral Test cannot be applied to the series

Let $f(x) = e^{1-x}$.

$f(x)$ is positive, continuous, and decreasing on $[1,\infty)$, so the Integral Test can be applied

$$\int_1^{\infty} e^{1-x}\,dx = \lim_{t\to\infty} \int_1^t e^{1-x}\,dx = \lim_{t\to\infty} -e^{1-x}\Big|_1^t$$

$$= \lim_{t\to\infty}\left[-e^{1-t} + e^0\right]$$

$$= 0 + 1 = 1$$

Because $\int_1^{\infty} f(x)\,dx$ converges, $\sum_{n=1}^{\infty} e^{1-n}$ converges by the Integral Test

5. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges and that its sum is 3. Which of the following statements must be true?

X I. $\lim_{n\to\infty} a_n = 3$

✓ II. $\lim_{n\to\infty} a_n = 0$

✓ III. $\lim_{N\to\infty} S_N = 3$

X IV. $\lim_{N\to\infty} S_N = 0$

(A) I and III only (B) I and IV only (C) II and III only (D) II and IV only (E) None of these

6. Which of the following statements is true concerning the series $\sum_{n=1}^{\infty} \frac{n!e^n}{n^n}$?

(A) The Ratio Test yields 0, so the series converges

(B) The Ratio Test yields $\frac{1}{e}$, so the series converges

(C) The Ratio Test yields 1, so the test is inconclusive

(D) The Ratio Test yields e, so the series diverges

(E) The Ratio Test yields ∞, so the series diverges

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,e^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,e^n}\right| = \lim_{n\to\infty}\frac{(n+1)\cdot e\cdot n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} e\cdot\left(\frac{n}{n+1}\right)^n$$

$$= e\cdot 1/e$$

$$= 1 \Rightarrow \text{Ratio Test is inconclusive}$$

7. Find the sum of the series $\sum_{n=1}^{\infty} \frac{-5+3^n}{2^{2n}}$.

(A) $\frac{3}{7}$ (B) $\frac{4}{3}$ (C) $\frac{7}{3}$ (D) $\frac{14}{3}$ (E) The series diverges

$$\sum_{n=1}^{\infty} \frac{-5+3^n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{-5+3^n}{4^n} = \sum_{n=1}^{\infty} -5\cdot\left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$$

$$= \frac{-5\cdot 1/4}{1-1/4} + \frac{3/4}{1-3/4}$$

$$= \frac{-5/4}{3/4} + \frac{3/4}{1/4}$$

$$= -\frac{5}{3} + 3 = \frac{4}{3}$$

8. Which of the following statements is true concerning the three series below?

$$\mathbf{A}: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad \mathbf{B}: \sum_{n=1}^{\infty} (-1)^{n+1}\sqrt{\frac{n}{n+1}} \qquad \mathbf{C}: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n^3+1}$$

(A) **A** converges absolutely, **B** converges conditionally, and **C** diverges
(B) **A** converges conditionally, **B** converges absolutely, and **C** diverges
(C) **A** diverges, **B** converges absolutely, and **C** converges conditionally
(D) **A** converges absolutely, **B** diverges, and **C** converges conditionally
(E) **A** converges conditionally, **B** diverges, and **C** converges absolutely

A: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges by AST, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series Test

$\Rightarrow$ converges conditionally

B: $\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}\sqrt{\frac{n}{n+1}}$ diverges by Test for Divergence

C: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n^3+1}$ converges by AST, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+1}$ converges by LCT

(compare to $b_n = \frac{1}{n^{5/2}}$)

$\Rightarrow$ converges absolutely

9. Which of the following statements is true concerning the series $\sum_{n=1}^{\infty} \frac{5-\sin n}{n^5}$?

(A) The series diverges by the Direct Comparison Test when comparing with $\sum_{n=1}^{\infty} \frac{4}{n^5}$

(B) The series diverges by the Direct Comparison Test when comparing with $\sum_{n=1}^{\infty} \frac{6}{n^5}$

(C) The series converges by the Direct Comparison Test when comparing with $\sum_{n=1}^{\infty} \frac{4}{n^5}$

(D) The series converges by the Direct Comparison Test when comparing with $\sum_{n=1}^{\infty} \frac{6}{n^5}$

(E) The Direct Comparison Test cannot be used to determine the convergence of the series

$-1 \leq -\sin n \leq 1$

$4 \leq 5-\sin n \leq 6$

$\frac{4}{n^5} \leq \frac{5-\sin n}{n^5} \leq \frac{6}{n^5}$

Choose $b_n = \frac{6}{n^5}$. $\sum_{n=1}^{\infty} b_n$ is a convergent p-series with $p=5>1$

Because $\sum b_n$ converges and $a_n \leq b_n$, $\sum a_n$ con.

10. How many of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n}$ $p=1 \Rightarrow$ Diverge

II. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $p=1, q=1 \Rightarrow$ Diverge

III. $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ $p=2 \Rightarrow$ Converge

IV. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ $p=1, q=2 \Rightarrow$ Converge

$\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$

converges: $p>1$ or $p=1$ and $q>1$

diverges: $p<1$ or $p=1$ and $q \leq 1$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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Part II: Free Response There are **4** questions on this portion of the exam. Show **ALL** work clearly in the space provided for each problem, unless the problem says otherwise. Your work must be complete, logical, and understandable or it will receive no credit. Please cross out or fully erase any work that you do not want graded. A total of **55** points are available on this portion of the exam.

FR Scores	
1	/10
2	/15
3	/15
4	/15
FR Total	/55

1. Find the sum of the series $\sum_{n=3}^{\infty} [\arctan(n-1) - \arctan(n+1)]$.

$$S_N = \sum_{n=3}^{N} [\arctan(n-1) - \arctan(n+1)] = \arctan 2 - \cancel{\arctan 4}$$

$$+ \arctan 3 - \cancel{\arctan 5}$$

$$+ \cancel{\arctan 4} - \cancel{\arctan 6}$$

$$\vdots$$

$$+ \cancel{\arctan(N-2)} - \arctan N$$

$$+ \cancel{\arctan(N-1)} - \arctan(N+1)$$

$$S_N = \arctan 2 + \arctan 3 - \arctan N - \arctan(N+1)$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} [\arctan 2 + \arctan 3 - \arctan N - \arctan(N+1)]$$

$$= \arctan 2 + \arctan 3 - \pi/2 - \pi/2$$

$$= \boxed{\arctan 2 + \arctan 3 - \pi}$$

2. (*a*) Use the Direct Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^3-1}}$ converges or diverges.

$$a_n = \frac{n}{\sqrt{4n^3-1}} \geq \frac{n}{\sqrt{4n^3}} = \frac{n}{2n^{3/2}} = \frac{1}{2n^{1/2}} = b_n$$

$\sum_{n=1}^{\infty} b_n$ is a divergent p-series with $p = 1/2 < 1$

Because $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$, $\sum_{n=1}^{\infty} a_n$ diverges by the Direct Comparison Test.

(*b*) Use the Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{8n^2+3n}{7^n(n^2-n+7)}$ converges or diverges.

$$a_n = \frac{8n^2+3n}{7^n(n^2-n+7)} \sim \frac{8n^2}{7^n \cdot n^2} \sim \frac{8}{7^n} \Rightarrow \text{Choose } b_n = \frac{1}{7^n}.$$

$\sum_{n=1}^{\infty} b_n$ is a convergent geometric series with $|r| = \frac{1}{7} < 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{8n^2+3n}{7^n(n^2-n+7)}}{\frac{1}{7^n}} = \lim_{n\to\infty} \frac{8n^2+3n}{n^2-n+7} = 8$$

Because $\sum_{n=1}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test

3. (*a*) Use the Alternating Series Test to show the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{1/n}}{n}$ converges. Be sure to check that the conditions of the Alternating Series Test are satisfied.

$b_n = \frac{e^{1/n}}{n}$ Three conditions: 1. $b_n \geq 0$ ✓
2. $\{b_n\}$ is decreasing ✓
3. $\lim_{n\to\infty} b_n = 0$ ✓

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{1/n}}{n}$ converges by the Alternating Series Test

(*b*) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{1/n}}{n}$ converge absolutely or conditionally? Justify your answer.

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}e^{1/n}}{n}\right| = \sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ Let $a_n = \frac{e^{1/n}}{n}$ and $b_n = \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series Test with $p=1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{e^{1/n}}{n}}{1/n} = \lim_{n\to\infty} e^{1/n} = e^0 = 1$

Because $\sum_{n=1}^{\infty} b_n$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges by the Limit Comparison Test. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{1/n}}{n}$ converges conditionally

(*c*) What is the maximum possible error of using the first 3 terms to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{1/n}}{n}$?

Maximum possible error of using S_3 to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{1/n}}{n}$ $= b_4 = \boxed{\frac{e^{1/4}}{4}}$

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}$ converges or diverges. Be sure to show all of your work.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}(3[n+1]-2)}{\cancel{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}(2[n+1]-1)} \cdot \frac{\cancel{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}}{\cancel{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}}$$

$$= \lim_{n\to\infty} \frac{3n+1}{2n+1}$$

$$= \frac{3}{2} > 1$$

Because $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, $\sum_{n=1}^{\infty} a_n$ diverges by the Ratio Test.