MAC2312 Exam 2

Thursday, October 28, 2021 3:52 PM

1. TFD? $\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ div

a) $\sum \frac{\ln n}{n}$ $\qquad \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ $\qquad (\ln n)^q \ll n^p$

b) $\sum \ln\left(\frac{3n^2+1}{4n^2+n+1}\right)$ $\qquad \lim\limits_{n\to\infty} \ln\left(\frac{3n^2 \sim}{4n^2 \sim}\right) = \ln\left(\frac{3}{4}\right) \neq 0$

$$\lim_{n\to\infty} \frac{an^b}{cn^d} \qquad \begin{array}{l} b>d \Rightarrow \text{DNE} \\ b=d \Rightarrow \frac{a}{c} \\ b<d \Rightarrow 0 \end{array}$$

c) $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ $\qquad \lim\limits_{n\to\infty} \frac{1}{n} - \frac{1}{n+1} = 0 - 0 = 0$

d) $\sum \frac{1}{2-\cos\frac{1}{n}}$ $\qquad \lim\limits_{n\to\infty} \frac{1}{2-\cos\frac{1}{n}} = \frac{1}{2-\cos 0} = \frac{1}{2-1} = 1 \neq 0$

2. converge <u>absolutely</u>?

i) $\sum \frac{(-1)^n e^n}{\sqrt{n!}}$ $\qquad$ ~~ii)~~ $\sum \frac{2-\sin(n!)}{\sqrt[3]{n}}$

ratio

i) $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ $\qquad \lim\limits_{n\to\infty} \frac{e^{n+1}}{\cancel{e^n}} \cdot \frac{\cancel{\sqrt{n!}}}{\sqrt{(n+1)!}} = \lim\limits_{n\to\infty} \frac{e}{\sqrt{n+1}} = 0 < 1$

$e^{n+1} = \cancel{e^n} \cdot e^1$

$\sqrt{(n+1)!} = \sqrt{(n+1)} \cdot \cancel{\sqrt{n!}}$

ii) $-1 \leq \sin(n!) \leq 1$

$$\boxed{\frac{1}{\sqrt[3]{n}} \leq \frac{2-\sin(n!)}{\sqrt[3]{n}}} \leq \frac{3}{\sqrt[3]{n}} \qquad p = \frac{1}{3}$$

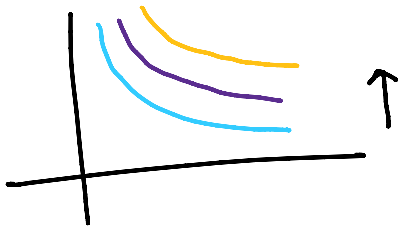
$$\boxed{\frac{1}{\sqrt[3]{n}} \le \frac{2-\sin(n!)}{\sqrt[3]{n}}} \le \frac{3}{\sqrt[3]{n}}$$

$p = \frac{1}{3}$

$\sum \frac{1}{\sqrt[3]{n}}$ div

p-series:

$\sum \frac{1}{n^p} \quad p \le 1 \Rightarrow$ div



③ monotonic inc. & bounded

$a_1 = \frac{1}{2} \qquad a_n = \sqrt{1+a_{n-1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L$$

$L > \frac{1}{2}$

$$\lim_{n\to\infty}\left(a_n = \sqrt{1+a_{n-1}}\right)$$

$$L = \sqrt{1+L} \Rightarrow L^2 = 1+L \Rightarrow L^2 - L - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \frac{1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{\frac{1+\sqrt{5}}{2}} \qquad \cancel{\frac{1-\sqrt{5}}{2}}$$

$a_n = \{a_1, a_2, a_3 \ldots\}$

$\lim_{n\to\infty} a_n = c$ to conv.

$\sum a_n = a_1 + a_2 + a_3 + \cdots a_n$

$\lim_{n\to\infty} a_n = 0$ to conv.

④ P. $\sum_{n=5}^{\infty} \frac{1}{n^3-n}$ $\quad$ ~~q.~~ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\quad$ R. $\sum_{n=2}^{\infty} \frac{\sqrt{3}}{e^n \pi}$

sum: geo & tele

$$\downarrow \qquad\qquad \searrow$$

$$\sum ar^n \qquad\qquad \sum a_n - a_{n-1}$$

P. $\dfrac{1}{n^3-n} \Rightarrow \dfrac{1}{n(n+1)(n-1)} = \dfrac{A}{n} + \dfrac{B}{n+1} + \dfrac{C}{n-1}$

$$\underset{\sim\sim}{n=0} \qquad \underset{\sim\sim}{n=-1} \qquad \underset{\sim\sim}{n=1}$$

$$\frac{1}{1(-1)} = A = -1 \qquad \frac{1}{-1(-2)} = B = \frac{1}{2} \qquad \frac{1}{1(2)} = C = \frac{1}{2}$$

$$\frac{1}{2}\left(\frac{1}{n-1}\right) - \frac{1}{n} + \frac{1}{2}\left(\frac{1}{n+1}\right)$$

$$\frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n} - \frac{1}{n} + \frac{1}{n+1}\right]$$

$n=5 \quad \frac{1}{2}\left[\frac{1}{4} - \cancel{\frac{1}{5}} \quad - \frac{1}{5} + \cancel{\frac{1}{6}}\right]$

$n=6 \quad \frac{1}{2}\left[\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} \quad -\cancel{\frac{1}{6}} + \cancel{\frac{1}{7}}\right]$

$\vdots \qquad\qquad \vdots$

$n=N \quad \frac{1}{2}\left[\cancel{\frac{1}{N-1}} - \frac{1}{N} \quad -\cancel{\frac{1}{N}} + \frac{1}{N+1}\right]$

$$\Rightarrow \lim_{N\to\infty} \frac{1}{2}\left[\frac{1}{4} - \frac{1}{5} - \cancel{\frac{1}{N}}^{\,0} + \cancel{\frac{1}{N+1}}^{\,0}\right] = \left[\frac{1}{20}\right]\frac{1}{2}$$

$$= \boxed{\frac{1}{40}}$$

R) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{3}}{e^n \pi} \rightarrow \sum_{n=2}^{\infty} \left(\frac{\sqrt{3}}{\pi}\right)\left(\frac{1}{e}\right)^n \qquad \sum_{n=c}^{\infty} ar^n \quad S = \frac{\text{first}}{1-r} = \frac{ar^c}{1-r}$

$$\frac{\frac{\sqrt{3}}{\pi}\left(\frac{1}{e}\right)^2}{\frac{e}{e} - \frac{1}{e}} = \frac{\frac{\sqrt{3}}{\pi e^{\cancel{2}}}}{\frac{e-1}{\cancel{e}}} = \boxed{\frac{\sqrt{3}}{\pi e(e-1)}}$$

(5)

$$\sum \frac{n^{1/4}}{\sqrt{n^k + 9n}} \qquad\qquad \sum \frac{1}{n^p} \quad p > 1 \Rightarrow \text{conv}$$

$$\sum \frac{n^{1/4}}{n^{k/2}} \qquad k \quad 1 \quad 4 \qquad k \in \left(\frac{5}{2}, \infty\right)$$

$\sum \frac{n^{1/4}}{n^{k/2}}$ $\quad \frac{k}{2} - \frac{1}{4} > \frac{4}{4}$ $\quad k \in (\frac{5}{2}, \infty)$

$k > \frac{5}{2}$

⑥ A. $\sum \left(\frac{k}{k+1}\right)^{k^2}$ B. $\sum \frac{(k!)^2}{(2k)!}$

root ratio

A $\lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{k+1}\right)^{k^2}} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e} < 1$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

B $\lim_{k\to\infty} \frac{((k+1)!)^2}{\cancel{(k!)^2}} \cdot \frac{\cancel{(2k)!}}{(2(k+1))!} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$

$(k+1)\cancel{k!} \cdot (k+1) \cdot \cancel{k!}$

$(2k+2)!$

$(2k+2)(2k+1)\cancel{(2k)!}$

$k^2 + 2k + 1$

$4k^2 + 6k + 2$

⑦
- ~~ratio $= 1 \Rightarrow$ inconclusive~~
- ~~TFD $\Rightarrow$ only diverence~~
- root $< 1 \Rightarrow$ converge
- ~~root $= 1 \Rightarrow$ inconclusive~~

⑧ ~~A.~~ $\sum_{n=10}^{\infty} \tan\frac{1}{n}$ B $\sum_{n=10}^{\infty} \sin^3\left(\frac{1}{n}\right)$

$\lim_{♡\to 0} \frac{\sin ♡}{♡} = 1$ (tan ♡)

A. $\sum \tan\frac{1}{n} \sim \sum \frac{1}{n} \Rightarrow$ div p-test $p=1$

$$\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$$

B $\sum \sin^3\left(\frac{1}{n}\right) \sim \sum \frac{1}{n}\cdot\frac{1}{n}\cdot\frac{1}{n} = \sum \frac{1}{n^3} \Rightarrow$ conv $p=3$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

9 $\sum_{n=1}^{\infty} (-1)^n b_n$

a) $\frac{1}{n\ln n}$ ~~b)~~ $\frac{2^n}{e^n}$ ~~c)~~ $\frac{(n+1)!}{n^n}$

a) $\sum \frac{1}{n^p(\ln n)^q}$

$p>1 \Rightarrow$ conv
$p=1$ $q>1 \Rightarrow$ conv
$p=1$ $q\leq 1 \Rightarrow$ div
$p<1 \Rightarrow$ div

integral: $\int_1^{\infty} \frac{1}{x\ln x}\,dx$ $u=\ln x$ $du=\frac{1}{x}\,dx$

$\int_0^{\infty} \frac{1}{u}\,du = \ln(\ln x) \rightarrow$ div ✓

AST $\rightarrow (-1)^n$ & $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} \frac{1}{n\ln n} = 0 \quad ✓$$

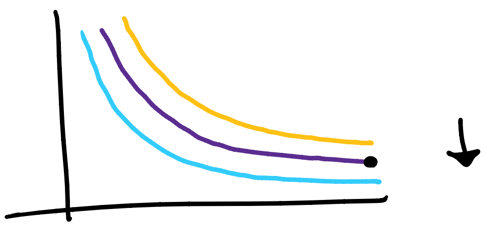
b) $\frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n \rightarrow r<1 \Rightarrow$ conv. abs.
geo.

c) $\frac{(n+1)!}{n^n}$ ratio conv. abs.

$$\lim_{n\to\infty} \frac{\overset{(n+2)\cancel{(n+1)}!}{(n+2)!}}{\cancel{(n+1)!}} \cdot \frac{n^n}{\underset{(n+1)^n\cdot(n+1)}{(n+1)^{n+1}}} = \lim_{n\to\infty} \frac{n+2}{n+1}\left(\frac{n}{n+1}\right)^n = 1\cdot\frac{1}{e} = \frac{1}{e} < 1$$

10. $0 \le h(x) \le f(x) \le g(x)$ $\quad x \ge 1$

$\sum f(n)$ conv



$\int f(x)\,dx \Rightarrow$ conv ✓

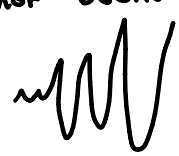
$\int h(x)\,dx \Rightarrow$ conv ✗

$\int g(x)\,dx \Rightarrow$ conv ✓

Free Response

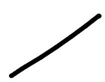
1. a) neither mono nor bounded

$(-1)^n n$



b) monotonic but unbounded

n

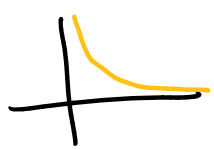


c) not mono. but bounded

$\cos(n)$ $\quad (-1)^n$

d) mono & bounded

$\frac{1}{n}$



2. $$\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} + \frac{5}{4^n}$$

tele. $\quad$ geo

geo. & tele.

$$\sum_{n=2}^{\infty} \frac{1}{\underset{(n+2)(n+3)}{n^2+5n+6}} + \sum_{n=2}^{\infty} \frac{5}{4^n}$$

$\hookrightarrow \sum_{n=2}^{\infty} 5\left(\frac{1}{4}\right)^n$

$S = \frac{5\left(\frac{1}{4}\right)^2}{} \cdot \frac{5}{4^{\cancel{2}}}$

$(n+2)(n+3)$

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

$n=-2$ $\qquad$ $n=-3$

$\frac{1}{1} = A = 1$ $\qquad$ $\frac{1}{-1} = B = -1$

$$\frac{1}{n+2} - \frac{1}{n+3}$$

$n=2$ $\quad \frac{1}{4} - \cancel{\frac{1}{5}}$

$n=3$ $\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$\vdots$

$n=N$ $\quad \cancel{\frac{1}{N+2}} - \frac{1}{N+3}$

$$\lim_{N\to\infty} \frac{1}{4} - \frac{1}{N+3} = \frac{1}{4}$$

$\hookrightarrow \sum_{n=2} 5\left(\frac{1}{4}\right)^n$

$$S = \frac{5\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}} = \frac{\frac{5}{4^{\cancel{2}}}}{\frac{3}{\cancel{4}}}$$

$$\frac{5}{12}$$

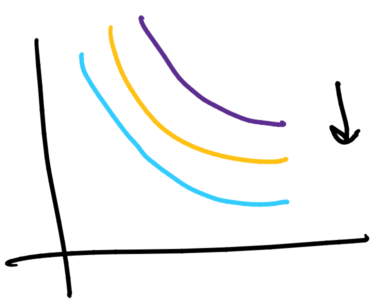
$$S = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \boxed{\frac{2}{3}}$$

3 a) $\sum_{n=10}^{\infty} \frac{10-\sin n}{n^4} \rightarrow$ $\boxed{\text{converge by DCT}}$

$-1 \le \sin n \le 1$

$$\frac{9}{n^4} \le \boxed{\frac{10-\sin n}{n^4} \le \frac{11}{n^4}}$$

p-test $p=4$
$\hookrightarrow$ conv



$\sum \frac{11}{n^4}$ conv. by p-test

$\lim_{n\to\infty} \sin n = \text{DNE}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2+3^n}$ $\qquad$ add S_4

$$|\text{error } S_N| \le a_{N+1}$$

$$\boxed{\frac{1}{2+3^5}}$$

$$a_5 \ge |\text{error } S_4|$$

(4) $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$ ratio $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty} \frac{(n+1)!}{\cancel{n!}} \cdot \frac{\cancel{1\cdot 3\cdot 5\cdots(2n-1)}}{\cancel{1\cdot 3\cdot 5\cdots(2n-1)}(2(n+1)-1)} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

$(n+1)!= (n+1)\cancel{n!}$

$(2n+2-1)$

$\boxed{\text{converges}}$

$(2n+1)!$

$(2(n+1)+1)!$

$1\cdot 4\cdot 7\cdots(3n-2)(3n+1)$

$+3$ $+3$ $+3$

Multiple choice (Part 2)

(1) ratio inconclusive? $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

a) $\sum \frac{e^n}{5^n}$ b) $\frac{1\cdot 3\cdot 5\cdots(2n+1)}{n^n}$ c. $\frac{n!}{e^{n^2}}$

d.) $\sum \frac{n^2}{n^7+1}$ e) $\frac{n}{(n+1)!}$

$(\ln n)^q \ll n^p \ll \; | \; c^n \ll n! \ll n^n$

inconclusive | ratio

a) $\lim\limits_{n\to\infty} \dfrac{\overset{\cancel{e^n}\cdot e}{e^{n+1}}}{\cancel{e^n}} \cdot \dfrac{\cancel{5^n}}{\underset{\cancel{5^n}\cdot 5}{5^{n+1}}} = \dfrac{e}{5} < 1$

b) $\lim\limits_{n\to\infty} \dfrac{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}(2n+3)}{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}} \cdot \dfrac{n^n}{\underset{(n+1)^n\cdot(n+1)}{(n+1)^{n+1}}} = \lim\limits_{n\to\infty} \dfrac{2n+3}{n+1}\left(\dfrac{n}{n+1}\right)^n = 2\dfrac{1}{e}$

$= \dfrac{2}{e} < 1$

c) $\lim\limits_{n\to\infty} \dfrac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{\cancel{n!}} \cdot \dfrac{\cancel{e^{n^2}}}{\underset{\cancel{e^{n^2}}\cdot e^{2n+1}}{e^{(n+1)^2}}} = \lim\limits_{n\to\infty} \dfrac{n+1}{e^{2n+1}} = 0$

$\sum \dfrac{n^2}{n^7+1}$

d) $\lim\limits_{n\to\infty} \dfrac{\overset{n^2\sim}{(n+1)^2}}{n^2} \cdot \dfrac{n^7+1}{\underset{n^7\sim}{(n+1)^7+1}} = 1 \Rightarrow$ inconclusive

e) $\lim\limits_{n\to\infty} \dfrac{n+1}{n} \cdot \dfrac{\cancel{(n+1)!}}{\underset{(n+2)\cancel{(n+1)!}}{(n+2)!}} = \lim\limits_{n\to\infty} \dfrac{n+1}{n(n+2)} = 0$

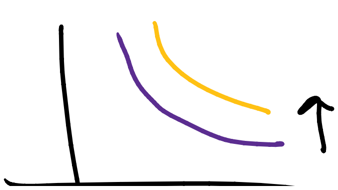
2) $\sum\limits_{n=5}^{\infty} \dfrac{3-(-1)^n}{n}$

$-1 \le (-1)^n \le 1$

$\boxed{\dfrac{2}{n} \le \dfrac{3-(-1)^n}{n}} \le \dfrac{4}{n}$

p-test $p=1$

$\sum \dfrac{2}{n}$ div



DCT $\rightarrow$ div

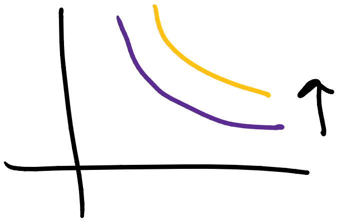
$n=5$ ..

$-1 \le (-1) \le 1$

$\boxed{\frac{2}{n} \le \frac{3-(-1)^n}{n}} \le \frac{4}{n}$

p-test $p=1$

$\sum \frac{2}{n}$ div



DCT $\rightarrow$ div

(3) $\sum_{n=5}^{\infty} \frac{(-1)^n}{\sqrt{n}\,\ln n}$

$\sum \frac{1}{\sqrt{n}\,\ln n} \sim \sum \frac{1}{\sqrt{n}\cdot\sqrt{n}} = \sum \frac{1}{n} \Rightarrow$ div

$\frac{1}{\sqrt{n}\,\ln n} > \frac{1}{\sqrt{n}\cdot\sqrt{n}}$

AST

$\lim_{n\to\infty} \frac{1}{\sqrt{n}\,\ln n} = 0$ conv.

diverges by DCT

$\boxed{\text{Converges conditionally}}$

(4) partial $\sum_{n=1}^{N} a_N = 3 - \frac{1}{N!}$ $\quad a_3 + a_4 = ?$

$\sum_{n=1}^{4} a_N = a_1 + a_2 + a_3 + a_4$

$\sum_{n=1}^{2} a_N = a_1 + a_2$

$\cancel{3} - \frac{1}{4!} - \left(\cancel{3} - \frac{1}{2!}\right) = \frac{1}{2!} - \frac{1}{4!}$

$4\cdot3\cdot2\cdot1$

$$\frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

$$230 - 220 \times 0.5 = 5!$$