

① $\sum_{n=1}^{\infty} a_n$ $\lim_{n \rightarrow \infty} a_n \neq 0 \rightarrow \text{div}$

a) $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$ ✗

b) $\lim_{n \rightarrow \infty} \ln\left(\frac{3n+1}{4n+1}\right) = \ln\left(\frac{3}{4}\right)$

c) $\lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$ ✗

d) $\lim_{n \rightarrow \infty} \frac{1}{2 - \cos\left(\frac{1}{n}\right)} = \frac{1}{2 - \cos 0} = \frac{1}{1} = 1$

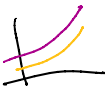
e

② conv. abs. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n}$

$\sum_{n=1}^{\infty} (-1)^n b_n$ $b_n \leftarrow \text{conv}$

i) $\frac{e^n}{\sqrt{n!}} \rightarrow \lim_{n \rightarrow \infty} \frac{e^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{e^n} = \frac{e}{\sqrt{n+1}} = 0 < 1$
conv

~~ii)~~ $\frac{1}{n^{1/5}} \leq \frac{2 - \sin(n!)}{n^{1/5}} \leq \frac{3}{n^{1/5}}$

$-1 \leq \sin x \leq 1$ 

d

③ $a_n = \sqrt{1 + a_{n-1}}$ $a_1 = \frac{1}{2}$

$L = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n-1}$

$L = \sqrt{1 + L}$

$L^2 = 1 + L$

$L^2 - L - 1 = 0$

$\frac{-(-1) \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

$\frac{1 + \sqrt{5}}{2}$ d

④ telescoping, geometric, alternating

P) $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

$\frac{1}{n(n-1)(n+1)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+1}$

$A = -1$ $B = \frac{1}{2}$ $C = \frac{1}{2}$

$\left(\frac{1}{2} \frac{1}{n-1} - \frac{1}{n} + \frac{1}{2} \frac{1}{n+1}\right)$

$\frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{2} \left(-\frac{1}{n} + \frac{1}{n+1}\right)$

~~Q)~~ $\sum \frac{1}{n^3}$

R) $\sum_{n=0}^{\infty} \frac{\sqrt{3}}{2^n \pi} \rightarrow \frac{\sqrt{3}}{\pi} \left(\frac{1}{2}\right)^n$

$$\textcircled{R)} \sum \frac{\sqrt[3]{3}}{e^{\frac{3}{\pi}}} \rightarrow \frac{\sqrt[3]{3}}{\pi} \left(\frac{1}{e}\right)^n$$

b

$$\textcircled{5} \sum \frac{n^{1/4}}{\sqrt{n^2 + 9n}} \quad p > 1$$

$$\frac{n^{1/4}}{n^{1/2}} = \frac{1}{n^{1/4}} \quad \times$$

$$\frac{n^{1/4}}{n^{k/2}} = \frac{1}{n^p} \quad \begin{matrix} p > 1 \\ \frac{k}{2} = \frac{1}{4} > \frac{1}{5} \\ k > \frac{5}{2} \end{matrix}$$

a

$$\textcircled{6} \textcircled{A)} \sum \left(\frac{k}{k+1}\right)^{k^2}$$

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} < 1$$

$$\lim_{k \rightarrow \infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e} < 1 \quad \checkmark$$

$$\boxed{\lim_{n \rightarrow \infty} \left(\frac{n}{n+a}\right)^n = \frac{1}{e^a}}$$

$$\textcircled{B)} \sum \frac{(k!)^2}{(2k)!}$$

$$\lim_{k \rightarrow \infty} \frac{[(k+1)!]^2}{[2(k+1)]!} \cdot \frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1 \quad \checkmark$$

d

$$\textcircled{7} \textcircled{i)} \lim \frac{a_{n+1}}{a_n} = 1$$

$$\textcircled{ii)} \lim a_n = 0$$

$$\textcircled{iii)} \lim a_n^{1/n} = 0 \quad \lim \sqrt[n]{a_n} < 1$$

$$\textcircled{iv)} \lim a_n^{1/n} = 1$$

b

8

$$\lim_{\psi \rightarrow 0} \frac{\sin \psi}{\psi} = 1 \quad \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

~~A)~~ $\sum \tan\left(\frac{1}{n}\right) = \sum \frac{\sin \frac{1}{n}}{\cos \frac{1}{n}}$ Div $\frac{\sin \frac{1}{n}}{\frac{1}{n}}$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{\cos \frac{1}{n}} = \frac{1}{\cos 0} = 1$$

$$\sum \frac{1}{n} \text{ div}$$

B)

$$\sum \sin^3 \frac{1}{n} = \sum \sin \frac{1}{n} \cdot \sin \frac{1}{n} \cdot \sin \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n} \cdot \sin \frac{1}{n} \cdot \sin \frac{1}{n}}{\frac{1}{n}} = 1$$

$$\sum \frac{1}{n^3} \text{ conv}$$

c

9

not absolutely $\sum b_n$ div

but $\sum (-1)^n b_n$ conv

a)

$$\frac{1}{n \ln n}$$

$$\int_1^{\infty} \frac{1}{n \ln n} (dn) \quad u = \ln n \quad du = \frac{1}{n} dn$$

$$\int_1^{\infty} \frac{1}{u} du = \ln(\ln n) \checkmark$$

alternating

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0 \checkmark$$

$$\frac{1}{n \ln n} > \frac{1}{(n+1) \ln(n+1)} \checkmark$$

$$\frac{1}{n^p}$$

~~b)~~ $b_n = \frac{2^n}{e^n} \rightarrow \left(\frac{2}{e}\right)^n \checkmark$
 $\frac{2}{e} < 1$

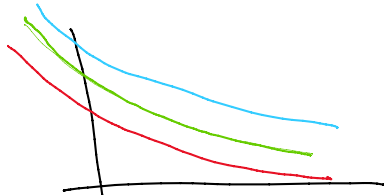
~~c)~~ $b_n = \frac{(n+1)!}{n^n}$

$$\lim_{n \rightarrow \infty} \frac{(n+2)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(n+1)!} = \frac{n+2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

a

10

$$0 \leq \underline{h(x)} \leq \underline{f(x)} \leq \underline{g(x)} \quad x \geq 1$$



$\sum \underline{f(n)}$ conv.

~~a)~~ $\int_1^{\infty} h(x) dx$

b) $\int_1^{\infty} f(x) dx$

c) $\int_1^{\infty} \underline{g(x)} dx$

c

Free Response 1

1

monotonic - goes in 1D

bounded - limit, a max or min value

a) $(-1)^n n$
 $(-1)^n e^n$
 $n \cos n$

b) $n e^n$

c) $\cos n \quad \sin n \quad (-1)^n$

d) $\frac{1}{n^2} \quad \frac{1}{e^n}$

2

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 5n + 6} + \frac{5}{4^n}$$

geo. telescoping

n=

2

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 5n + 6} + \frac{5}{4^2}$$

↑

$$\frac{1}{(n+3)(n+2)} = \frac{A}{n+3} + \frac{B}{n+2}$$

$$A = -1 \quad B = 1$$

$$-\frac{1}{n+3} + \frac{1}{n+2}$$

$$\begin{array}{l} n=2 \\ n=3 \\ n=4 \\ n=N \end{array} \quad \begin{array}{l} -\frac{1}{5} + \frac{1}{4} \\ -\frac{1}{6} + \frac{1}{5} \\ -\frac{1}{7} + \frac{1}{6} \\ -\frac{1}{N+3} + \frac{1}{N+2} \end{array}$$

geo. telescoping

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r} \quad \begin{array}{l} n= \\ 2 \quad 3 \quad 4 \dots \\ 2 \quad 3 \quad 4 \dots \end{array}$$

$$\sum_{n=1}^{\infty} 5 \left(\frac{1}{4}\right)^n = 5 \left(\frac{1}{4}\right)^1$$

$$\begin{array}{l} \frac{5 \cdot \frac{1}{4}}{3} - \frac{0}{5} \\ \frac{5}{3} - \frac{0}{5} = \frac{5}{3} \end{array}$$

$$\lim_{N \rightarrow \infty} \frac{1}{4} - \frac{1}{N+3} = \frac{1}{4}$$

$$S = \frac{1}{4} + \frac{5}{12} = \frac{8}{12} = \frac{2}{3}$$

3

a) $\sum_{n=10}^{\infty} \frac{10 - \sin n}{n^4}$

$$-1 \leq \sin n \leq 1$$

$$\sum \frac{11}{n^4} \text{ conv.}$$

$$\frac{9}{n^4} \leq \frac{10 - \sin n}{n^4} \leq \frac{11}{n^4}$$

$$\text{DCT} \rightarrow \sum \frac{10 - \sin n}{n^4} \text{ conv}$$

because

$$\sum \frac{11}{n^4} \text{ conv}$$

$$\& \frac{10 - \sin n}{n^4} \leq \frac{11}{n^4}$$

b)

$$\sum \frac{(-1)^n}{2+3^n} \quad \text{first 4 terms}$$

1 2 3 4 5

$$|\text{error}| \leq |a_{n+1}|$$

$$\left| \frac{(-1)^5}{2+3^5} \right| = \frac{1}{245}$$

$$\left| \frac{(-1)^n}{2+3^5} \right| = \frac{1}{245}$$

4

$$\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!} = \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

conv.

$$[1 \cdot 3 \cdot 5] \quad z(2)+1 = 5$$

$$(2-1)(2(2)-1)(2(3)-1)$$

$$(2n-1)(2n+1)$$

$$\frac{(2n-1)(2(n+1)-1)}{(2n+2-1) = (2n+1)}$$

1) ratio test inconclusive

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \rightarrow \text{inconclusive}$$

$$\text{A) } \lim_{n \rightarrow \infty} \frac{e^{n+1}}{5^{n+1}} \cdot \frac{5^n}{e^n} = \frac{e}{5} < 1 \text{ conv}$$

$$\text{B) } \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)(2n+3)}{(n+1)^{n+1}} \cdot \frac{n^n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} = \frac{2n+3}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e} \text{ conv}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+a} \right)^n = \frac{1}{e^a}$$

$$\text{C) } \lim_{n \rightarrow \infty} \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \frac{n+1}{e^{2n+1}} = 0 < 1$$

$$\text{D) } \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+1)^2+1} \cdot \frac{n^2}{n^2} = 1$$

$$\text{E) } \lim_{n \rightarrow \infty} \frac{n+1}{n^2+2n} \cdot \frac{(n+1)!}{n!} = \frac{n+1}{n^2+2n} = 0 < 1$$

$$\textcircled{1} \lim_{n \rightarrow \infty} \frac{n+1}{(n+2)!} \cdot \frac{(n+1)!}{n} = \frac{n+1}{(n+2)(n+1)!} = \frac{n+1}{n^2+2n} = 0 < 1$$

d

$$\textcircled{2} \sum_{n=5}^{\infty} \frac{3-(-1)^n}{n} \text{ div by DCT}$$

$$-1 \leq (-1)^n \leq 1$$

$$\frac{2}{n} \leq \frac{3-(-1)^n}{n} \leq \frac{4}{n}$$

$\sum \frac{2}{n}$ - div p-series

$$\textcircled{3} \sum_{n=5}^{\infty} \frac{(-1)^n}{\sqrt{n} \ln n}$$

$\frac{1}{\sqrt{n} \ln n}$ - div p-series

$$\frac{1}{\sqrt{n}} < \frac{1}{\ln n}$$

$$\frac{1}{n} = \frac{1}{\sqrt{n} \sqrt{n}} < \frac{1}{\sqrt{n} \ln n} \text{ DCT}$$

div

alternating test

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \ln n} = 0 \quad \checkmark$$

$$\frac{1}{\sqrt{n+1} \ln(n+1)} < \frac{1}{\sqrt{n} \ln n} \quad \checkmark$$

converges conditionally

④

$$S = \sum_{n=1}^{\infty} a_n \quad S_N = \sum_{n=1}^N a_n = 3 - \frac{1}{N!} \quad a_3 + a_4.$$

$$\sum_{n=1}^4 a_n = \cancel{a_1} + \cancel{a_2} + \underbrace{a_3 + a_4} = 3 - \frac{1}{4!}$$

$$\sum_{n=1}^2 a_n = a_1 + a_2 = 3 - \frac{1}{2!}$$

$$\sum_{n=1}^4 a_n - \sum_{n=1}^2 a_n = a_3 + a_4$$

$$\cancel{3} - \frac{1}{4!} - (\cancel{3} - \frac{1}{2!}) = \frac{12}{24} - \frac{1}{24} = \frac{11}{24}$$

b

$$\sum_{n=1}^N a_n = n$$

$$N=1 \quad a_1$$

$$\sum_{n=1}^2 \quad N=2 \quad a_1 + a_2$$

$$\sum_{n=1}^3 \quad N=3 \quad a_1 + a_2 + a_3$$

$$(n+1)! = (n+1) \cdot n!$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5!$$

TFD

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\sum \neq 0$$

~~A)~~ $S_N - S_{N-1} = a_N$

~~B)~~ $\sum \frac{1}{n} \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

~~C)~~

D)

i) $\pi^{1-2n} = \pi \left(\frac{1}{\pi^2}\right)^n$ R

ii) $\frac{\sin^2 n}{\sqrt{n+10}} \quad 0 \leq \frac{\sin^2 n}{\sqrt{n+10}} \leq \frac{1}{\sqrt{n+10}}$ P

iii) $\left(\frac{n-1}{n}\right)^{2n} = \left[\left(1 - \frac{1}{n}\right)^n\right]^2 = \left(\frac{1}{e}\right)^2$

AST

$\ll 1^2 \quad \pi$

AJ 1

9

Q $\sqrt{n} \frac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{n}} \leftarrow \sqrt{n} \cdot \left(\frac{\pi}{n}\right)^2 = \frac{\pi}{n^{3/2}} \leftarrow \text{conv}$

$\lim_{n \rightarrow \infty} \frac{\cancel{\sqrt{n}} \sin^2 \frac{\pi}{n}}{\cancel{\sqrt{n}} \left(\frac{\pi}{n}\right)^2} \cdot \frac{1}{\cancel{\cos \frac{\pi}{n}}} = 1$

R

$\sum_{n=5}^{\infty} \sin\left(\frac{\pi}{n^2}\right) \leftarrow \text{conv} \quad \frac{\pi}{n^2} \leftarrow \text{conv}$

$\lim_{n \rightarrow \infty} \frac{\sin \frac{\pi}{n^2}}{\frac{\pi}{n^2}} = 1$

$\sum (-1)^n a_n$

$0 \leq a_{n+1} \leq a_n$

$\lim_{n \rightarrow \infty} a_n = 0$

① TFD? $\Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n$ div

a) $\sum \frac{\ln n}{n}$ $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$ $(\ln n)^2 \ll n^p$

b) $\sum \ln\left(\frac{3n^2+1}{4n^2+n+1}\right)$ $\lim_{n \rightarrow \infty} \ln\left(\frac{3n^2 \sim}{4n^2 \sim}\right) = \ln\left(\frac{3}{4}\right) \neq 0$

$\lim_{n \rightarrow \infty} \frac{a n^b}{c n^d}$ $b > d \Rightarrow \text{DNE}$
 $b = d \Rightarrow \frac{a}{c}$
 $b < d \Rightarrow 0$

c) $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ $\lim_{n \rightarrow \infty} \frac{1}{n} - \frac{1}{n+1} = 0 - 0 = 0$

d) $\sum \frac{1}{2 - \cos \frac{1}{n}}$ $\lim_{n \rightarrow \infty} \frac{1}{2 - \cos \frac{1}{n}} = \frac{1}{2 - \cos 0} = \frac{1}{2-1} = 1 \neq 0$

② converge absolutely?

i) $\sum \frac{(-1)^n e^n}{\sqrt{n!}}$
ratio

~~ii) $\sum \frac{2 - \sin(n!)}{\sqrt[3]{n}}$~~

i) $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

$\lim_{n \rightarrow \infty} \frac{e^{\cancel{n}} \cdot e^1}{e^{\cancel{n}}} \cdot \frac{\cancel{\sqrt{n!}}}{\sqrt{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{e}{\sqrt{(n+1)} \cdot \cancel{\sqrt{n!}}} = \lim_{n \rightarrow \infty} \frac{e}{\sqrt{n+1}} = 0 < 1$

ii) $-1 \leq \sin(n!) \leq 1$

$\frac{1}{\sqrt[3]{n}} \leq \frac{2 - \sin(n!)}{\sqrt[3]{n}} \leq \frac{3}{\sqrt[3]{n}}$

$p = \frac{1}{3}$
 $\Rightarrow \perp \dots$

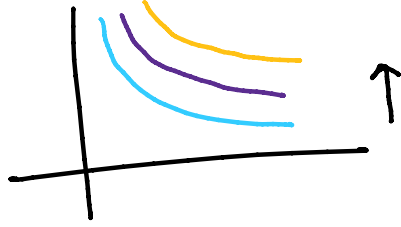
p-series:

$$\sum \frac{1}{n^p} \quad p \leq 1 \Rightarrow \text{div}$$

$$\frac{1}{\sqrt[3]{n}} = \frac{2 - \sin(\pi/n)}{\sqrt[3]{n}} = \frac{0}{\sqrt[3]{n}}$$

$$p = \frac{1}{3}$$

$$\sum \frac{1}{\sqrt[3]{n}} \text{ div}$$



③ monotonic inc. & bounded

$$a_1 = \frac{1}{2} \quad a_n = \sqrt{1+a_{n-1}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n-1} = L$$

$$L > \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} (a_n = \sqrt{1+a_{n-1}})$$

$$L = \sqrt{1+L} \Rightarrow L^2 = 1+L \Rightarrow L^2 - L - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1 + \sqrt{5}}{2} \quad \frac{1 - \sqrt{5}}{2}$$

$$a_n = \{a_1, a_2, a_3, \dots\}$$

$$\lim_{n \rightarrow \infty} a_n = C \text{ to conv.}$$

$$\sum a_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\lim_{n \rightarrow \infty} a_n = 0 \text{ to conv.}$$

④ P. $\sum_{n=5}^{\infty} \frac{1}{n^2-n}$

Q. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

R. $\sum_{n=2}^{\infty} \frac{\sqrt{3}}{e^{3n}\pi}$

Sum: geo & tele

$$\downarrow$$

$$\sum ar^n$$

$$\downarrow$$

$$\sum a_n - a_{n-1}$$

$$P. \frac{1}{n^2 - n} \Rightarrow \frac{1}{n(n+1)(n-1)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n-1}$$

$$\frac{1}{1(-1)} = A = -1$$

$$\frac{1}{-1(-2)} = B = \frac{1}{2}$$

$$\frac{1}{1(2)} = C = \frac{1}{2}$$

$$\frac{1}{2} \left(\frac{1}{n-1} \right) - \frac{1}{n} + \frac{1}{2} \left(\frac{1}{n+1} \right)$$

$$\frac{1}{2} \left[\frac{1}{n-1} - \frac{1}{n} - \frac{1}{n} + \frac{1}{n+1} \right]$$

$$n=5 \quad \frac{1}{2} \left[\frac{1}{4} - \frac{1}{5} - \frac{1}{5} + \frac{1}{6} \right]$$

$$n=6 \quad \frac{1}{2} \left[\frac{1}{5} - \frac{1}{6} - \frac{1}{6} + \frac{1}{7} \right]$$

$$\vdots$$

$$n=N \quad \frac{1}{2} \left[\frac{1}{N-1} - \frac{1}{N} - \frac{1}{N} + \frac{1}{N+1} \right]$$

$$\Rightarrow \lim_{N \rightarrow \infty} \frac{1}{2} \left[\frac{1}{4} - \frac{1}{5} - \frac{1}{N} + \frac{1}{N+1} \right] = \left[\frac{1}{20} \right] \frac{1}{2}$$

$$= \boxed{\frac{1}{40}}$$

$$R) \sum_{n=2}^{\infty} \frac{\sqrt{3}}{e^{2n} \pi} \rightarrow \sum_{n=2}^{\infty} \left(\frac{\sqrt{3}}{\pi} \right) \left(\frac{1}{e} \right)^n \quad \sum_{n=c}^{\infty} ar^n \quad S = \frac{\text{first}}{1-r} = \frac{ar^c}{1-r}$$

$$\frac{\frac{\sqrt{3}}{\pi} \left(\frac{1}{e} \right)^2}{e^2 - \frac{1}{e}} = \frac{\frac{\sqrt{3}}{\pi e^2}}{e^2 - \frac{1}{e}} = \boxed{\frac{\sqrt{3}}{\pi e(e-1)}}$$

5)

$$\sum \frac{n^{1/4}}{n^k + 9n}$$

$$\sum \frac{1}{n^p} \quad p > 1 \Rightarrow \text{conv}$$

$$\sum \frac{n^{1/4}}{n^{k/2}}$$

$$k \quad 1 \quad 4$$

$$k \in \left(\frac{5}{2}, \infty \right)$$

$$\sum \frac{n^{1/4}}{n^{k/2}}$$

$$\frac{k}{2} - \frac{1}{4} > \frac{1}{4}$$

$$k \in \left(\frac{5}{2}, \infty\right)$$

$$k > \frac{5}{2}$$

⑥ A: $\sum \left(\frac{k}{k+1}\right)^{k^2}$
root

B: $\sum \frac{(k!)^2}{(2k)!}$
ratio

A $\lim_{k \rightarrow \infty} \sqrt[k]{\left(\frac{k}{k+1}\right)^{k^2}} = \lim_{k \rightarrow \infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e} < 1$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

B $\lim_{k \rightarrow \infty} \frac{(k+1)k! \cdot (k+1) \cdot k!}{(k!)^2} \cdot \frac{(2k)!}{(2(k+1))!} = \lim_{k \rightarrow \infty} \frac{(k^2 + 2k + 1)(k+1)^2}{(2k+2)(2k+1)(2k)!} = \frac{1}{4} < 1$

- ⑦
- ~~ratio = 1~~ \Rightarrow inconclusive
 - ~~TFD~~ \Rightarrow only divergence
 - root < 1 \Rightarrow converge
 - ~~root = 1~~ \Rightarrow inconclusive

⑧ ~~Pr.~~ $\sum_{n=10}^{\infty} \tan \frac{1}{n}$

B $\sum_{n=10}^{\infty} \sin^3\left(\frac{1}{n}\right)$

$$\lim_{\heartsuit \rightarrow 0} \frac{\sin \heartsuit}{\heartsuit} = 1$$

A. $\sum \tan \frac{1}{n} \sim \sum \frac{1}{n} \Rightarrow$ div p-test $p=1$

$$\lim_{n \rightarrow \infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = 1$$

B $\sum \sin^3(\frac{1}{n}) \sim \sum \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} = \sum \frac{1}{n^3} \Rightarrow$ conv $p=3$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}} \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

9 $\sum_{n=1}^{\infty} (-1)^n b_n$

a) $\frac{1}{n \ln n}$ b) $\frac{2^n}{e^n}$ c) $\frac{(n+1)!}{n^n}$

a) $\sum \frac{1}{n^p (\ln n)^q}$

$p > 1 \Rightarrow$ conv
 $p = 1 \quad q > 1 \Rightarrow$ conv
 $p = 1 \quad q \leq 1 \Rightarrow$ div
 $p < 1 \Rightarrow$ div

integral: $\int \frac{1}{x \ln x} dx$

$u = \ln x$
 $du = \frac{1}{x} dx$

$\int \frac{1}{u} du = \ln(\ln x) \rightarrow$ div ✓

AST $\rightarrow (-1)^n \Delta \lim_{n \rightarrow \infty} b_n = 0$

$\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$ ✓

b) $\frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n \rightarrow r < 1 \Rightarrow$ conv. abs. geo.

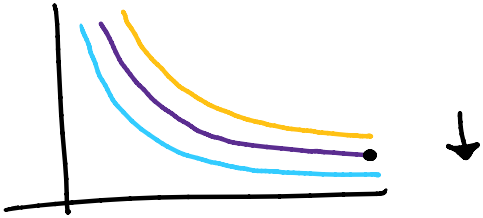
c) $\frac{(n+1)!}{n^n}$ ratio conv. abs.

$$\lim_{n \rightarrow \infty} \frac{(n+2)!}{(n+1)!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \frac{n+2}{n+1} \left(\frac{n}{n+1}\right)^n = 1 \cdot \frac{1}{e} = \frac{1}{e} < 1$$

10

$$0 \leq \underline{h(x)} \leq \underline{f(x)} \leq \underline{g(x)} \quad x \geq 1$$

$\sum f(n)$ conv



$$\int f(x) dx \Rightarrow \text{conv} \quad \checkmark$$

$$\int h(x) dx \Rightarrow \text{conv} \quad \times$$

$$\int g(x) dx \Rightarrow \text{conv} \quad \checkmark$$

Free Response

1 a) neither mono nor bounded

$$(-1)^n n \quad \text{wavy line}$$

b) monotonic but unbounded

$$n \quad \text{line}$$

c) not mono. but bounded

$$\cos(n) \quad (-1)^n$$

d) mono & bounded

$$\frac{1}{n} \quad \text{decreasing curve}$$

2

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 5n + 6} + \frac{5}{4^n}$$

tele. geo

geo. & tele.

$$\sum_{n=2}^{\infty} \frac{1}{(n+2)(n+3)} + \sum_{n=2}^{\infty} \frac{5}{4^n}$$

$$\sum_{n=2}^{\infty} \frac{5}{4^n} \rightarrow \sum_{n=2}^{\infty} 5 \left(\frac{1}{4}\right)^n = \frac{5 \left(\frac{1}{4}\right)^2}{\frac{1}{4}}$$

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

$$n = -2$$

$$\frac{1}{1} = A = 1$$

$$n = -3$$

$$\frac{1}{-1} = B = -1$$

$$\frac{1}{n+2} - \frac{1}{n+3}$$

$$n=2 \quad \frac{1}{4} - \frac{1}{5}$$

$$n=3 \quad \frac{1}{5} - \frac{1}{6}$$

$$\vdots$$

$$n=N \quad \frac{1}{N+2} - \frac{1}{N+3}$$

$$\lim_{N \rightarrow \infty} \frac{1}{4} - \frac{1}{N+3} = \frac{1}{4}$$

$$S = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

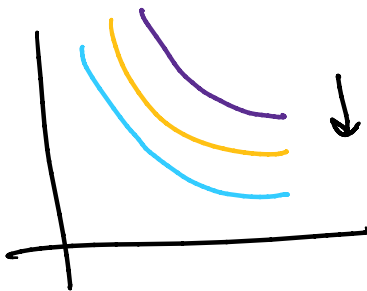
$$\hookrightarrow \sum_{n=2}^{\infty} 5 \left(\frac{1}{4}\right)^n$$

$$S = \frac{5 \left(\frac{1}{4}\right)^2}{1 - \frac{1}{4}} = \frac{5 \cdot \frac{1}{16}}{\frac{3}{4}} = \frac{5}{12}$$

③ a) $\sum_{n=10}^{\infty} \frac{10 - \sin n}{n^4} \rightarrow$ converge by DCT
 $-1 \leq \sin n \leq 1$

$$\frac{9}{n^4} \leq \frac{10 - \sin n}{n^4} \leq \frac{11}{n^4}$$

p-test $p=4$
 \hookrightarrow conv



$$\sum \frac{11}{n^4} \text{ conv. by p-test}$$

$$\lim_{n \rightarrow \infty} \sin n = \text{DNE}$$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2+3^n}$ add S_4

$$|\text{error } S_n| \leq a_{n+1}$$

$$\frac{1}{2+3^5}$$

$$a_5 \geq |\text{error } S_4|$$

④ $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ ratio $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\lim_{n \rightarrow \infty} \frac{\cancel{(n+1)!}}{\cancel{n!}} \cdot \frac{\cancel{1 \cdot 3 \cdot 5 \cdots (2n-1)}}{\cancel{1 \cdot 3 \cdot 5 \cdots (2n-1)}(2(n+1)-1)} = \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

converges

$$\frac{(2n+1)!}{(2(n+1)+1)!}$$

$$\frac{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)}{+3 \quad +3 \quad +3}$$

Multiple choice (Part 2)

① ratio inconclusive? $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

a) $\sum \frac{e^n}{5^n}$

b) $\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n^n}$

c) $\frac{n!}{e^{n^2}}$

d.) $\sum \frac{n^2}{n^2+1}$

e) $\frac{n}{(n+1)!}$

$$(\ln n)^2 \ll n^p \ll c^n \ll n! \ll n^n$$

inconclusive ratio

a) $\lim_{n \rightarrow \infty} \frac{e \cdot e}{e^{n+1}} \cdot \frac{5^n}{5^{n+1}} = \frac{e}{5} < 1$

b) $\lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \frac{2n+3}{n+1} \left(\frac{n}{n+1}\right)^n = 2 \frac{1}{e} = \frac{2}{e} < 1$

c) $\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{(n+1)^2}} = \lim_{n \rightarrow \infty} \frac{n+1}{e^{2n+1}} = 0$

$$\sum \frac{n^2}{n^{2+1}}$$

d) $\lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} \cdot \frac{n^7+1}{(n+1)^7+1} = 1 \Rightarrow \text{inconclusive}$

e) $\lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{(n+1)!}{(n+2)!} = \lim_{n \rightarrow \infty} \frac{n+1}{n(n+2)} = 0$

② $\sum_{n=5}^{\infty} \frac{3 - (-1)^n}{n}$

$$-1 \leq (-1)^n \leq 1$$

$$\frac{2}{n} \leq \frac{3 - (-1)^n}{n} \leq \frac{4}{n}$$

p-test $p=1$

$\sum \frac{2}{n}$ div



DCT \rightarrow div

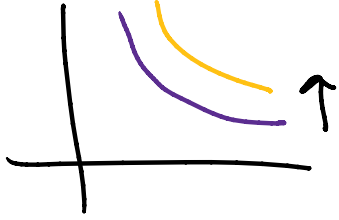
$n=5$..

$$-1 \leq (-1)^n \leq 1$$

p-test $p=1$

$$\frac{2}{n} \leq \frac{3 - (-1)^n}{n} \leq \frac{4}{n}$$

$\sum \frac{2}{n}$ div



DCT \rightarrow div

$$\textcircled{3} \sum_{n=5}^{\infty} \frac{(-1)^n}{\sqrt{n} \ln n}$$

$$\sum \frac{1}{\sqrt{n} \ln n} \sim \sum \frac{1}{\sqrt{n} \cdot \sqrt{n}} = \sum \frac{1}{n} \Rightarrow \text{div}$$

AST

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \ln n} = 0 \quad \text{Conv.}$$

$$\frac{1}{\sqrt{n} \ln n} > \frac{1}{\sqrt{n} \cdot \sqrt{n}}$$

\uparrow
diverges by DCT

Converges conditionally

$$\textcircled{4} \text{ partial } \sum_{n=1}^N a_n = 3 - \frac{1}{N!} \quad a_3 + a_4 = ?$$

$$\sum_{n=1}^4 a_n = a_1 + a_2 + a_3 + a_4$$

$$\sum_{n=1}^2 a_n = a_1 + a_2$$

$$\cancel{3} - \frac{1}{4!} - (\cancel{3} - \frac{1}{2!}) = \frac{1}{2!} - \frac{1}{4!}$$

\uparrow
4 · 3 · 2 · 1

$$\frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

$$230 - 220 \times 0.5 = 5!$$