Solutions

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① $\sum_{n=1}^{\infty} a_n \qquad \lim_{n\to\infty} a_n \neq 0 \rightarrow$ div

a) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ ✗

b) $\lim_{n\to\infty} \ln\left(\frac{3n^2+1}{4n^2+n+1}\right) = \ln\left(\frac{3}{4}\right)$

c) $\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$ ✗

d) $\lim_{n\to\infty} \frac{1}{2-\cos\frac{1}{n}} = \frac{1}{2-\cos 0} = \frac{1}{1} = 1$

$\uparrow$ 1

$\boxed{e}$

② conv. abs. $\quad \lim \frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n}$

$\sum_{n=1}^{\infty} (-1)^n b_n \qquad b_n \leftarrow$ conv

i) $\frac{e^n}{\sqrt{n!}} \rightarrow \lim_{n\to\infty} \frac{\cancel{e^{n+1}}\, e}{\sqrt{(n+1)!}} \cdot \frac{\cancel{\sqrt{n!}}}{\cancel{e^n}} = " \frac{e}{\sqrt{n+1}} = 0 < 1$

$\sqrt{(n+1)}\cancel{\sqrt{n!}}$

conv

~~ii)~~ $\frac{1}{n^{1/3}} \leq \frac{2-\sin(n!)}{n^{1/3}} \not\leq \frac{3}{n^{1/3}}$

$-1 \leq \sin x \leq 1$

$\boxed{d}$

③ $a_n = \sqrt{1+a_{n-1}} \qquad a_1 = \frac{1}{2}$

$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1}$

$L = \sqrt{1+L}$

$L^2 = 1 + L$

$L^2 - L - 1 = 0$

$\frac{-(-1) \pm \sqrt{1-4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

$\frac{1+\sqrt{5}}{2} \qquad \boxed{d}$

④ telescoping, geometric, alternating

P) $\sum_{n=6}^{\infty} \frac{1}{n^3-n} \qquad \frac{1}{n(n-1)(n+1)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+1}$

$A = -1 \qquad B = \frac{1}{2} \qquad C = \frac{1}{2}$

$-\frac{1}{2}\frac{1}{n} \quad \frac{1}{2}\frac{1}{n}$

$\left(\frac{1}{2}\frac{1}{n-1} - \frac{1}{n} + \frac{1}{2}\frac{1}{n+1}\right)$

$\frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{2}\left(-\frac{1}{n} + \frac{1}{n+1}\right)$

~~Q)~~ $\sum \frac{1}{n^3}$

R) $\sum \frac{\sqrt{3}}{e^n \pi} \rightarrow \frac{\sqrt{3}}{\pi}\left(\frac{1}{e}\right)^n$

(R) $\sum \frac{\sqrt{3}}{e^n \pi} \rightarrow \frac{\sqrt{3}}{\pi}\left(\frac{1}{e}\right)^n$

$\boxed{b}$

(5) $\sum \frac{n^{1/4}}{\sqrt{n^k + 9n}}$ $\quad p > 1$

$\frac{n^{1/4}}{n^{1/2}} = \frac{1}{n^{1/4}}$ ✗

$\frac{n^{1/4}}{n^{k/2}} = \frac{1}{n^p}$ $\quad p > 1$

$\frac{k}{2} - \frac{1}{4} > \frac{4}{4}$

$k > \frac{5}{2}$

$\boxed{a}$

(6) (A) $\sum \left(\frac{k}{k+1}\right)^{k^2}$

$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$

$\lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e} < 1$ ✓

$$\boxed{\lim_{n\to\infty} \left(\frac{n}{n+a}\right)^n = \frac{1}{e^a}}$$

(B) $\sum \frac{(k!)^2}{(2k)!}$

$$\lim_{k\to\infty} \frac{\overset{[(k+1)\cancel{(k!)}]^2}{[(k+1)!]^2}}{\underset{(2k+2)(2k+1)\cancel{(2k!)}}{[2(k+1)]!}} \cdot \frac{\cancel{(2k)!}}{\cancel{(k!)^2}} = \text{"} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$$ ✓

$\boxed{d}$

(7) ~~i)~~ $\lim \frac{a_{n+1}}{a_n} \overset{<}{=} 1$

~~ii)~~ $\lim a_n = 0$

iii) $\lim a_n^{1/n} = 0$ $\quad \lim \sqrt[n]{a_n} < 1$

~~iv)~~ $\lim a_n^{1/n} = 1$

$\boxed{b}$

8) $\lim_{♡\to 0} \frac{\sin ♡}{♡} = 1$ $\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$

$\lim_{n\to\infty} \frac{1}{n} = 0$

~~A)~~ $\sum \tan(\frac{1}{n}) = \sum \frac{\sin\frac{1}{n}}{\cos\frac{1}{n}}$ Div $\frac{\sin\frac{1}{n}}{\frac{1}{n}}$

$\lim_{n\to\infty} \underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{1} \cdot \frac{1}{\cos\underbrace{\tfrac{1}{n}}_{0}} = \frac{1}{\cos 0} = 1$

$\sum \frac{1}{n}$ div

B) $\sum \sin^3\frac{1}{n} = \sum \sin\frac{1}{n} \cdot \sin\frac{1}{n} \cdot \sin\frac{1}{n}$

$\lim \underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{1} \underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{1} \underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{1} = 1$

$\boxed{c}$ $\sum \frac{1}{n^3}$ conv

9) not absolutely $\underbrace{b_n \text{ div}}$

but $\underbrace{(-1)^n b_n \text{ conv}}$

a) $\frac{1}{n \ln n}$ $\int^\infty \frac{1}{n \ln n} dn$ $u = \ln n$, $du = \frac{1}{n} dn$

$\int^\infty \frac{1}{u} du = \ln(\ln n)$ ✓

alternating

$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ ✓ $\frac{1}{n^p}$

$\frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$ ✓

~~b)~~ $b_n = \frac{2^n}{e^n} \to (\frac{2}{e})^n$ ✓ <1

~~c)~~ $b_n = \frac{(n+1)!}{n^n}$ $\lim_{n\to\infty} \frac{(n+2)!}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{(n+1)!}} = " \underbrace{\frac{n+2}{n+1}}_{1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$

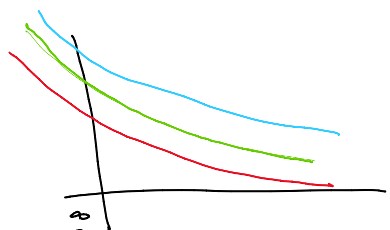
$(n+2)\cancel{(n+1)!}$

$(n+1)(n+1)^n$

$\boxed{a}$

(10) $0 \le h(x) \le f(x) \le g(x)$ $x \ge 1$

$\Sigma f(n)$ conv.



~~a)~~ $\int_1^\infty h(x)\,dx$

b) $\int_1^\infty f(x)\,dx$

c) $\int_1^\infty g(x)\,dx$

$\boxed{e}$

Free Response 1

(1) monotonic - goes in 1D

bounded - limit, a max or min value

a) $(-1)^n n$

$(-1)^n e^n$

$n \cos n$

b) n e^n

c) $\cos n$ $\sin n$ $(-1)^n$

d) $\frac{1}{n^2}$ $\frac{1}{e^n}$

(2) $\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} + \frac{5}{4^n}$ geo. telescoping

$n=$

(2) $\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} + \frac{5}{4^n}$

geo. telescoping

$$\frac{1}{(n+3)(n+2)} = \frac{A}{n+3} + \frac{B}{n+2}$$

$A=-1 \qquad B=1$

$$\frac{-1}{n+3} + \frac{1}{n+2}$$

$n=2 \quad \cancel{-\frac{1}{5}} + \frac{1}{4}$

$n=3 \quad \cancel{-\frac{1}{6}} + \cancel{\frac{1}{5}}$

$n=4 \quad \cancel{-\frac{1}{7}} + \cancel{\frac{1}{6}}$

$\vdots$

$n=N \quad -\frac{1}{N+3} + \cancel{\frac{1}{N+2}}$

$$\lim_{N\to\infty} \frac{1}{4} - \cancel{\frac{1}{N+3}}^{\,0} = \underbrace{\frac{1}{4}}$$

$$\underbrace{\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}}$$

$n=$ $\cancel{1}$ $\underbrace{2}$ 3 4 ...

2 3 4 ...

$\sum_{n=2}$

$$\left[\sum_{n=1}^{\infty} 5\left(\tfrac{1}{4}\right)^n\right] - 5\left(\tfrac{1}{4}\right)^1$$

$$\frac{5\cancel{\frac{1}{4}}}{\frac{3}{\cancel{4}}} - \frac{5}{4}$$

$$\frac{5}{3} - \frac{5}{4} = \frac{5}{12}$$

$$S = \frac{1}{4} + \frac{5}{12} = \frac{8}{12} = \boxed{\frac{2}{3}}$$

(3)

a) $\sum_{n=10}^{\infty} \frac{10-\sin n}{n^4}$

$\sum \frac{11}{n^4}$ conv.

$-1 \le \sin n \le 1$

$$\frac{9}{n^4} \le \frac{10-\sin n}{n^4} \le \frac{11}{n^4}$$

DCT $\rightarrow$ $\sum \frac{10-\sin n}{n^4}$ conv

because

$\sum \frac{11}{n^4}$ conv

& $\frac{10-\sin n}{n^4} \le \frac{11}{n^4}$

b) $\sum \frac{(-1)^n}{2+3^n}$ first $\underbrace{4}$ terms

1 2 3 4 $\underbrace{5}$

$|\text{error}| \le |a_{n+1}|$

$$\left|\frac{\cancel{(-1)^5}}{2+3^5}\right| = \frac{1}{245}$$

$$\left| \frac{\cancel{(-1)^5}}{2+3^5} \right| = \frac{1}{245}$$

④ $$\sum_{n=1}^{\infty} \frac{n!}{1\cdot3\cdot5\cdots(2n-1)}$$

$$\lim_{n\to\infty} \frac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2n+1)} \cdot \frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}}{\cancel{n!}} = \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

conv.

$[1\cdot3\cdot5]$ $\quad 2(2)+1 = 5$

$$\frac{\begin{array}{c}(2-1)\,(2(2)-1)\,(2(3)-1) \\ (2n-1)\,(2n+1)\end{array}}{\begin{array}{c}(2n-1)\,(2(n+1)-1) \\ (2n+2-1) = (2n+1)\end{array}}$$

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① ratio test inconclusive

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \rightarrow \text{inconclusive}$$

A) $$\lim_{n\to\infty} \frac{\overset{\cancel{e^n}\cdot e}{e^{n+1}}}{\underset{\cancel{5^n}\cdot 5}{5^{n+1}}} \cdot \frac{\cancel{5^n}}{\cancel{e^n}} = \frac{e}{5} < 1 \quad \text{conv}$$

B) $$\lim_{n\to\infty} \frac{\cancel{1\cdot3\cdot5\cdots(2n+1)}(2n+3)}{\underset{(n+1)^1(n+1)^n}{(n+1)^{n+1}}} \cdot \frac{n^n}{\cancel{1\cdot3\cdot5\cdots(2n+1)}} = \frac{2n+3}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e} \quad \text{conv}$$

$$\boxed{\lim_{n\to\infty}\left(\frac{n}{n+a}\right)^n = \frac{1}{e^a}}$$

C) $$\lim_{n\to\infty} \frac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{e^{(n+1)^2}} \cdot \frac{\cancel{e^{n^2}}}{\cancel{n!}} = \frac{n+1}{e^{2n+1}} = 0 < 1$$

e^{n^2+2n+1}

$\cancel{e^{n^2}} \cdot e^{2n+1}$

D) $$\lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^7+1} \cdot \frac{n^7+1}{n^2} = 1$$

E) $$\lim \frac{n+1}{\ \ ,\ } \cdot \frac{\cancel{(n+1)!}}{} = \frac{n+1}{n^2+2n} = 0 < 1$$

$$\cancel{E.)}\ \lim_{n\to\infty} \frac{n+1}{(n+2)!}\cdot\frac{\cancel{(n+1)!}}{n} = " \frac{n+1}{n^2+2n} = 0 < 1$$

$(n+2)\cancel{(n+1)!}$

$\boxed{d}$

(2) $\displaystyle\sum_{n=5}^{\infty} \frac{3-(-1)^n}{n}$ div by DCT

$-1 \le (-1)^n \le 1$

$$\underbrace{\frac{2}{n} \le \frac{3-(-1)^n}{n}} \le \frac{4}{n}$$

$\sum \frac{2}{n}$ - div p-series

(3) $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^n}{\sqrt{n}\,\ln n}$

$\frac{1}{\sqrt{n}\ln n}$ - div p-series

$$\frac{1}{\sqrt{n}} < \frac{1}{\ln n}$$

$$\frac{1}{n} = \frac{1}{\sqrt{n}\sqrt{n}} < \frac{1}{\sqrt{n}\ln n} \quad \text{DCT}$$

div

alternating test

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}\ln n} = 0$$

$$\frac{1}{\sqrt{n+1}\,\ln(n+1)} < \frac{1}{\sqrt{n}\ln n}$$

$\boxed{\text{converges conditionally}}$

4

$$S = \sum_{n=1}^{\infty} a_n \qquad S_N = \sum_{n=1}^{N} a_n = 3 - \frac{1}{N!} \qquad a_3 + a_4.$$

$$\sum_{n=1}^{4} a_n = \cancel{a_1 + a_2} + \underbrace{a_3 + a_4} = 3 - \frac{1}{4!}$$

$$\sum_{n=1}^{2} a_n = a_1 + a_2 = 3 - \frac{1}{2!}$$

$$\sum_{n=1}^{4} a_n - \sum_{n=1}^{2} a_n = a_3 + a_4$$

$$\cancel{3} - \frac{1}{4!} - \left(\cancel{3} - \frac{1}{2!}\right) = \frac{12}{24} - \frac{1}{24} = \frac{11}{24}$$

$\boxed{b}$

$$\sum_{n=1}^{N} a_n = n$$

$\sum^{2}$ $\sum^{3}$

$N=1 \quad a_1$

$N=2 \quad a_1 + a_2.$

$N=3 \quad a_1 + a_2 + a_3$

$(n+1)! = (n+1) \cdot n!$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5!$

TFD

$\lim_{n \to \infty} a_n = 0$

$\sum \neq 0$

~~A)~~ $\underbrace{S_N - S_{N-1} = a_N}$

~~B)~~ $\sum \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$

~~C)~~

D)

i) $\pi^{1-2n} = \pi \left(\frac{1}{\pi^2}\right)^n$ $\boxed{R}$

ii) $\frac{\sin^2 n}{\sqrt{n+10}}$ $0 \leq \frac{\sin^2 n}{\sqrt{n+10}} \leq \frac{1}{\sqrt{n+10}}$ $\boxed{P}$

iii) $\left(\frac{n-1}{n}\right)^{2n}$ $\left(\frac{n}{n} - \frac{1}{n}\right)^{2n} = \left[\left(1 - \frac{1}{n}\right)^n\right]^2$

$\left(\frac{1}{e}\right)^2$

AST

$(-1)^2$ π

AOI

9) Q) $\sqrt{n}\,\dfrac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{n}} \leftarrow \sqrt{n} \cdot \left(\frac{\pi}{n}\right)^2 = \frac{\pi}{n^{3/2}}$ ↖ conv

$$\lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\ \cancel{\sin^2 \frac{\pi}{n}}^{\,1}}{\cancel{\sqrt{n}}\ \cancel{\left(\frac{\pi}{n}\right)^2}} \cdot \frac{1}{\cancel{\cos \frac{\pi}{n}}^{\,1}} = 1$$

$\frac{\pi}{n} \to 0$

R) $\sum_{n=5}^{\infty} \sin\left(\frac{\pi}{n^2}\right)$ ↖ conv $\qquad \frac{\pi}{n^2} \leftarrow$ conv

$$\lim_{n\to\infty} \frac{\sin \frac{\pi}{n^2}}{\frac{\pi}{n^2}} = 1$$

$\sum (-1)^n a_n \qquad 0 \le a_{n+1} \le a_n$

$$\lim_{n\to\infty} a_n = 0$$