Exam 2 Review Sheet

Thursday, March 11, 2021 2:34 PM

sequences

series types

- Test for Divergence (TFD)
- Geometric
- Telescoping
- Direct Comparison Test (DCT)
- Limit Comparison Test (LCT)
- p-series (not a test)
- Integral
- Alternating
- Root
- Ratio

* Be careful of the lower bound of the summation

$$\sum_{n=1}^{\infty} a_n$$

sequence

$$\{a_1, a_2, a_3, a_4 \dots\}$$

convergent?

↳ $\lim_{n \to \infty} a_n = L$

Limit Laws

$\lim_{n \to \infty} (a_n) = A$ & $\lim_{n \to \infty} (b_n) = B$

1. $\lim_{n \to \infty} (a_n \pm b_n) = A \pm B$

2. $\lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B$

3. $\lim_{n \to \infty} (c \cdot b_n) = c \cdot B$ (c is a constant)

4. $\lim_{n \to \infty} (a_n / b_n) = \frac{A}{B}$

5. $\lim_{n \to \infty} (a_n)^p = A^p$ $p > 0, a_n > 0$

$\ln(n)^q << n^p << c^n << n! << n^n$

squeeze thm

if the limit of the max & min of a sequence are equal, its limit is that #

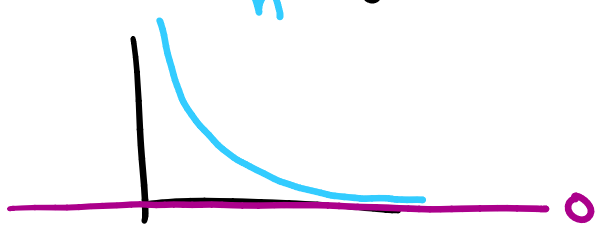
ex. $\left\{ \frac{\cos^2 n}{2^n} \right\}$ $0 \leq \cos^2 n \leq 1$

$$\frac{0}{2^n} \leq \frac{\cos^2 n}{2^n} < \frac{1}{2^n} \qquad \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$0 \leq \lim_{n\to\infty} \frac{\cos^2 n}{2^n} \leq 0$$

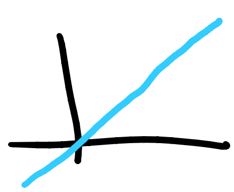
Bounded: is restricted by a number

ex. $\frac{1}{n^5}$ bounded below by 0



monotonic: moves in only one direction

ex n is monotonic



Every bounded monotonic sequence is convergent

Recursive sequence

$a_1 = c \qquad a_{n+1} = f(n)$

convergent ?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$$

ex. monotonic, decreasing

$a_1 = 2 \qquad a_{n+1} = \frac{1}{3-a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3-a_n}$$

$$\Downarrow$$

$$L = \frac{1}{3-L} \Rightarrow (3-L)L = 1$$

$$L^2 - 3L + 1 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \frac{3 \pm \sqrt{9-4}}{2}$$

$$\cancel{\frac{3+\sqrt{5}}{2}}, \quad \boxed{\frac{3-\sqrt{5}}{2}}$$

$$0 \le a_n \le 2$$

Series

Test for divergence (TFD)

if $\lim_{n\to\infty} a_n \neq 0$

↳ divergent

if $\lim_{n\to\infty} a_n = 0$

↳ you know NOTHING

* I suggest always doing this first, even just a little mental check

Geometric

sum

$$\sum_{n=c}^{\infty} ar^n = \frac{ar^c}{1-r}$$

if $|r| < 1$

↳ convergent

if $|r| \geq 1$

↳ divergent

* look for a number to the n^{th} term

Telescoping

$$S_N = a_1 + a_2 + a_3 \cdots a_N$$

1. use partial fraction decomposion or log properties to make $a_n = a_n - a_{n+1}$ or something similar

2. write out sum & cancel

3. Find $\lim_{n\to\infty} S_N$

ex. $$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \frac{-1}{n+3} + \frac{1}{n+1}$$

$$\frac{2}{(n+3)(n+1)} = \frac{A}{(n+3)} + \frac{B}{(n+1)}$$

$n=-1$ $\qquad$ $n=-3$

$\frac{2}{2} = B$ $\qquad$ $\frac{2}{-2} = A$

$B=1$ $\qquad$ $A=-1$

$$S_N = a_1 = \cancel{-\frac{1}{4}} + \frac{1}{2}$$
$$+ a_2 = \cancel{-\frac{1}{5}} + \frac{1}{3}$$
$$+ a_3 = \cancel{-\frac{1}{6}} + \cancel{\frac{1}{4}}$$
$$\vdots \qquad \vdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

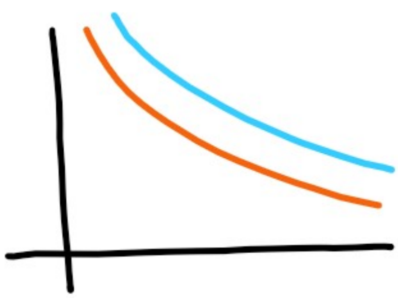
$$+\ a_{N-1} = \frac{-1}{N+2} + \cancel{\frac{1}{N}}$$

$$+\ a_N = \frac{-1}{N+3} + \cancel{\frac{1}{N+1}}$$

$$\lim_{n\to\infty} \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{N+2}}^{\,0} - \cancel{\frac{1}{N+3}}^{\,0} = \boxed{\frac{5}{6}}$$

Direct Comparison Test (DCT)

$$\sum_{n=1}^{\infty} a_n \quad \& \quad \sum_{n=1}^{\infty} b_n$$



$\sum b_n$ conv. ?

- ☐ $0 \le b_n \le a_n$
- ☐ a_n conv.

$\sum a_n$ div. ?

- ☐ $0 \le b_n \le a_n$
- ☐ b_n div.

a_n pushes b_n down to conv.
or
b_n pushes a_n up to div.

* look for a sum similar to something you <u>know</u>
like $\frac{1}{n-1}$ or $\frac{1}{n^2+1}$

Limit Comparison Test

$\sum_{n=1}^{\infty} a_n$ $\quad$ a_n & b_n are similar

conv.

- ☐ $\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$
- ☐ $\sum b_n$ conv

div.

- ☐ $\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$
- ☐ $\sum b_n$ div.

* look for similar summations and $\sin(\frac{1}{n})$ or $\tan(\frac{1}{n})$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

conv.
- ☐ $p > 1$

div.
- ☐ $p \leq 1$

not a real test... some TAs will accept it, others won't

Integral Test

$$\sum_{n=1}^{\infty} a_n$$

$a_n = f(n) \geq 0$

conv.

- [ ] $\int_1^{\infty} f(x)\,dx$ conv

div.

- [ ] $\int_1^{\infty} f(x)\,dx$ div.

* look for familiar integrals:
u-sub, IBP n's or ln n's
↳ can prove p-test

Alternating Series Test

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

conv.

- [ ] $0 < a_{n+1} \leq a_n$ or $\frac{da_n}{dn} < 0$

- [ ] $\lim_{n \to \infty} a_n = 0$

conditional conv.

$\sum_{n=1}^{\infty} (-1) a_n$ conv. but

$\sum_{n=1}^{\infty} a_n$ div.

absolute conv.

$\sum_{n=1}^{\infty} (-1) a_n$ conv. and

$\sum_{n=1}^{\infty} a_n$ conv.

Remainder:

$|R_N| \leq a_{N+1}$

* look for $(-1)^n$ multiplied by the entire sum

Root

$\sum_{n=1}^{\infty} a_n$

conv.

- [ ] $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1$

div.

- [ ] $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1$

unknown

☐ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

$\sqrt[n]{\ln(n)^{g}}$ & $\sqrt[n]{n^{p}} = 1$

* look for something to the n^{th} power

Ratio

$\sum_{n=1}^{\infty} a_n$

conv.

☐ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

div

☐ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

unknown

☐ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

* look for $1\cdot3\cdot5\cdots(2n-1)$, $n!$, or other such functions

Limits to know:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\tan\left(\frac{1}{n}\right) \sim \frac{1}{n}$$

$$\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$$

$$\lim_{n\to 0}\left(\frac{\sin \heartsuit}{\heartsuit}\right) = 1$$

$$\ln n \ll n^p \ll c^n \ll n! \ll n^n$$

when you get your exam

write test @ top:

TFD, Geo, Tele, Int.,

DCT, LCT, Alt., Root, Ratio

p-series

Examples:

① $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

Alt.

- [x] decreasing
- [x] $\lim_{n\to\infty} a_n = 0$

$$\frac{da_n}{dn} = \frac{n^2+n-\overbrace{(n-1)(2n+1)}^{2n^2-2n+n-1}}{(n^2+n)^2}$$

$$= \frac{-n^2+2n+1}{(n^2+n)^2}$$

neg → dec.

$$\lim_{n\to\infty} \frac{n-1}{n^2+n} = 0$$

bigger on bottom

$\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ LCT w/ $\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ div.

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^2+n}}{\frac{n}{n^2}} = \lim_{n\to\infty} \frac{n-1}{n} \cdot \frac{n^2}{n^2+1} = 1$$

div.

Conditional convergent

② $\sum_{n=1}^{\infty} \frac{2^n n!}{(2n+2)!}$ ratio

$$\lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n+2)!}{(\underbrace{2(n+1)+2}_{2n+2+2})!}$$

$$= \lim_{n\to\infty} \frac{2\cdot\cancel{2^n}}{\cancel{2^n}} \cdot \frac{(n+1)\cdot\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{(2n+2)!}}{(2n+4)(2n+3)\cancel{(2n+2)!}}$$

$$= \lim_{n\to\infty} \frac{2n+2}{(2n+4)(2n+3)} = 0$$

$\boxed{\text{conv.}}$

③ $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$ root

$$\lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{3n}{1+8n}\right)^{\cancel{n}}} = \frac{3}{8}$$

$\boxed{\text{conv.}}$

④ $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n}$

LCT w/ $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad \int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{3x^3}\Big|_1^b$

$= 0 - \left(-\frac{1}{3}\right)$ conv.

$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \quad \cancel{\frac{\frac{1}{n}}{\frac{1}{n}}}^{1} = 1$ ✓

$\boxed{\text{conv}}$

⑤ $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ root

$\lim_{n\to\infty} \sqrt[\cancel{n}]{\frac{(n!)^{\cancel{n}}}{n^{4\cancel{n}}}} = \lim_{n\to\infty} \frac{n!}{n^4} = \text{DNE}$

$\boxed{\text{div.}}$

⑥ $\sum_{n=2}^{\infty} \frac{5}{3^n}$ geo.

$\frac{5\left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} = \frac{\frac{5}{9}}{\frac{2}{3}} = \frac{5}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{2} = \boxed{\frac{5}{6}}$

$\boxed{\text{Conv.}}$