Part I: Multiple Choice

1. Which of the following series can we conclude diverges using the Test for Divergence?

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \ln\left(\frac{3n^2+1}{4n^2+n+1}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2-\cos\left(\frac{1}{n}\right)}$

   (e) (b) and (d)

   Ans: (a) and (c) both have $\lim_{n\to\infty} a_n = 0$; (b): $\lim_{n\to\infty} a_n = \ln\frac{3}{4} \neq 0$ (d) $\lim_{n\to\infty} a_n = 1 \neq 0$

2. Which of the following series converge absolutely?

   $i)$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n e^n}{\sqrt{n!}}$ $\qquad$ $ii)$ $\displaystyle\sum_{n=2}^{\infty} \frac{2-\sin(n!)}{\sqrt[3]{n}}$

   (a) Neither i or ii (b) Both i and ii (c) Only ii (d) Only i

   Ans.:(i) Ratio test, $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = 0 < 1$; (ii) Divergent by DCT with $b_n = \frac{1}{\sqrt[3]{n}}$

3. Given that the following sequence is monotonically increasing and bounded above, find the limit of the sequence.

   $$a_n = \sqrt{1+a_{n-1}},\ a_1 = \frac{1}{2}$$

   (a) 1 (b) $\sqrt{2}$ (c) 5 (d) $\frac{1+\sqrt{5}}{2}$ (e) Diverges by TFD

   Ans.: $L = \sqrt{1+L} \Longrightarrow L^2 - L - 1 = 0$

4. Which of the following series' sums may be found using techniques from this class?

P. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^3 - n}$ Q. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ R. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{3}}{e^n \pi}$

(a) P, Q, and R (b) P and R (c) P and Q (d) P only (e) Q only

Ans.: P. Telescoping; R: Geo;

5. Determine the values of k for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/4}}{\sqrt{n^k + 9n}}$ will converge.

(a) $k \in \left(\frac{5}{2}, \infty\right)$

(b) $k \in \left[\frac{5}{2}, \infty\right)$

(c) $k \in (3, \infty)$

(d) $k \in [3, \infty)$

(e) $k \in \left(\frac{8}{3}, \infty\right)$

Ans.: $\frac{k}{2} - \frac{1}{4} > 1 \Longrightarrow k > \frac{5}{2}$

6. Which of the statements below is true for the following two series?

$$A = \sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2} \qquad B = \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

(a) A converges and B diverges.

(b) A diverges and B converges.

(c) Both diverge.

(d) Both converge.

(e) The convergence of these series cannot be determined.

Ans.: A: Root Test, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{e} < 1$, Converges; B: Ratio test, $\lim\limits_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \frac{1}{4} < 1$, Converges.

7. How many of the following statements are true for a series $\sum a_n$ with positive terms?

- If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then the series converges conditionally.
- If $\lim\limits_{n\to\infty} a_n = 0$, then the series converges.
- If $\lim\limits_{n\to\infty} (a_n)^{\frac{1}{n}} = 0$, then the series converges absolutely.
- If $\lim\limits_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$, then the series diverges.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Ans.:

8. Which of the series below are convergent using the limit comparison test?

A. $\sum_{n=10}^{\infty} \tan\left(\frac{1}{n}\right)$

B. $\sum_{n=10}^{\infty} \sin^3\left(\frac{1}{n}\right)$

(a) Both A and B
(b) A only
(c) B only
(d) Neither A nor B
(e) The limit comparison test is inconclusive for these series.

Ans.: A: LCT, $b_n = \frac{1}{n}$, Diverges; B: LCT, $b_n = \frac{1}{n^3}$, Converges.

9. For which of the following sequences b_n would the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converge conditionally (and not absolutely)?

(a) $b_n = \frac{1}{n \ln n}$
(b) $b_n = \frac{2^n}{e^n}$ (Geo. series)
(c) $b_n = \frac{(n+1)!}{n^n}$ (Ratio test)
(d) (a), (b), and (c)
(e) None of the above

Ans.:

10. Suppose that $0 \leq h(x) \leq f(x) \leq g(x)$ for all $x \geq 1$ and all of the functions are continuous and decreasing. Which of the following would guarantee that $\sum_{n=1}^{\infty} f(n)$ converges?

(a) $\int_1^{\infty} h(x)\, dx$ converges
(b) $\int_1^{\infty} f(x)\, dx$ converges
(c) $\int_1^{\infty} g(x)\, dx$ converges
(d) (a), (b), and (c)
(e) (b) and (c)

Name: ______________________________

Part II: Free Response

FR Scores	
1	/4
2	/6
3	6 /4
4	4 /6
FR Total	/20

1. Give examples of the following (you do not need to justify your answers):

(a) A sequence that is neither monotonic nor bounded.

$\{(-1)^n e^n\}$

(b) A sequence that is monotonic but unbounded

$\{e^n\}$

(c) A sequence that is not monotonic but is bounded

$$\left\{(-1)^n \frac{1}{e^n}\right\}$$

(d) A sequence that is monotonic and bounded.

$$\left\{\frac{1}{e^n}\right\}$$

2. Find the sum of the following series, or if it diverges, state that it diverges. You may leave your answer as a sum of fractions.

$$\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} + \frac{5}{4^n}.$$

Show ALL work to receive full credits

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} = \sum_{n=2}^{\infty} \frac{1}{n+2} - \frac{1}{n+3}$

$$S_N = \frac{1}{4} - \frac{1}{5}$$

$$\frac{1}{5} - \frac{1}{6}$$

$$\vdots$$

$$\frac{1}{N+2} - \frac{1}{N+3}$$

$$S_N = \frac{1}{4} - \frac{1}{N+3}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} = \lim_{n\to\infty} S_N = \frac{1}{4}$$

(2) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{4^n} = \frac{\frac{5}{4^2}}{1-\frac{1}{4}} = \frac{5}{12}$ (Geometric series)

$$\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6} + \frac{5}{4^n} = \frac{1}{4} + \frac{5}{12} = \frac{2}{3}$$

3. **Show ALL work to receive full credits**.

(a) Determine if the series $\displaystyle\sum_{n=10}^{\infty} \frac{10-\sin n}{n^4}$ is convergent or divergent. Clearly indicate what test you are using to justify your conclusion and state why it applies.

$$\text{Since} \quad 0 < a_n = \frac{10-\sin n}{n^4} \leq \frac{11}{n^4} = b_n$$

$$\text{and} \quad \sum_{n=10}^{\infty} b_n \text{ Converges (by the } p\text{-test, } p = 4 > 1)$$

$$\Longrightarrow \sum_{n=10}^{\infty} a_n \text{ Converges (by the DCT)}$$

Circle one: (Convergent, Divergent), by DCT Test.

(b) Suppose that you choose to estimate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2+3^n}$ by adding the first 4 terms together. According to the Alternating Series Error Estimation Theorem, how large can the error of doing this estimate possibly be?

$$b_n = \frac{1}{2+3^n}$$

$$|s - S_4| \leq b_5 = \frac{1}{2+3^5} = \left(\frac{1}{245}\right)$$

4. Determine whether the following series converges or diverges. Clearly indicate what test you are using to justify your conclusion and state why it applies.

$$1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \ldots + \frac{n!}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} + \ldots$$

Show ALL work to receive full credits.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)! \quad 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! \quad 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{1}{2} < 1$$

$\Longrightarrow$ the series converges by the Ratio test.

Version B:

#2 : $\lim_{n\to\infty} S_N = 1$; Sum$= 1 + \frac{7}{20} = \frac{27}{20}$

#3(a) :$b_n = \frac{a}{n^2}$ where the constant a can be any number ≥ 9. $\sum b_n$ converges by $p-$test, $p = 2 > 1$, $\sum a_n$ converges by DCT.

#3(b) :$|\text{error}| \leq \frac{1}{5^3 + 3 = 128}\frac{1}{128}$.

Version C:

#2 : $\lim_{n\to\infty} S_N = \frac{1}{5}$; Sum$= \frac{1}{5} + \frac{9}{2} = \frac{47}{10}$

#3(a) :$b_n = \frac{a}{n^3}$ where the constant a can be any number ≥ 4. $\sum b_n$ converges by $p-$test, $p = 3 > 1$, $\sum a_n$ converges by DCT.

#3(b) :$|\text{error}| \leq \frac{1}{42}$.

Version D:

#2 : $\lim_{n\to\infty} S_N = \frac{1}{4}$; Sum$= \frac{1}{4} + \frac{3}{10} = \frac{11}{26}$

#3(a) :$b_n = \frac{a}{n^3}$ where the constant a can be any number ≥ 6. $\sum b_n$ converges by $p-$test, $p = 3 > 1$, $\sum a_n$ converges by DCT.

#3(b) : $|\text{error}| \leq \frac{1}{74}$.

Multiple Choices:

	Ver A	Ver B	Ver C	Ver D
Q1	E	A	D	A
Q2	D	C	E	E
Q3	D	D	C	B
Q4	B	E	E	A
Q5	A	D	E	B
Q6	D	C	D	E
Q7	B	C	B	C
Q8	C	B	C	D
Q9	A	C	A	C
Q10	E	B	B	C

	Ver. A	**Ver. B**	**Ver. C**	**Ver. D**
1.	d	e	d	d

2. (a) DCT

2. (c) diverges

3. (a) DCT ($|a_n|$ with $\dfrac{1}{\sqrt{n}\sqrt{n}}$ to show the series is NOT absolute convergent); AST (to show the series is conditionally convergent)

3. (c) converges conditionally

4.	b	a	a	b

2. Determine if the series $\sum_{n=5}^{\infty} \frac{3-(-1)^n}{n}$ converges or diverges.

(a) Which test did you use? DCT

(b) Show your work in determining whether the series converges or diverges.

$$b_n = \frac{2}{n} < \frac{3-(-1)^n}{n} = a_n,$$

and $\sum b_n = \sum \frac{1}{n}$ diverges by $p-$ series test $(p=1)$

$$\Rightarrow \quad \sum a_n = \sum \frac{3-(-1)^n}{n} \quad \text{also diverges}$$

(c) The series (converges, diverges). (circle one of the choices)

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3. Determine if the series $\sum_{n=5}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges absolutely, ,conditionally or diverges.

(a) Which test(s) did you use? **DCT (to show it does not converge absolutely)**,

AST (to show it converges conditionally.)

(b) Show your work.

1. $\sqrt{n}\ln n < \sqrt{n}\cdot\sqrt{n}$ for all $n \geq 5$, $\quad \Rightarrow \quad |a_n| = \frac{1}{\sqrt{n}\ln n} > \frac{1}{\sqrt{n}\cdot\sqrt{n}} = \frac{1}{n}$

   Since $\quad \sum \frac{1}{n}$ diverges $(p=1)$ $\quad \Rightarrow \quad \sum |a_n|$ diverges (by DCT)

2. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}\ln n} = 0$ and $\frac{1}{\sqrt{n}\ln n} = 0$ is decreasing

   $\Rightarrow \quad \sum \frac{(-1)^n}{\sqrt{n}\ln n}$ converges (by AST)

   (1) and (2) $\Rightarrow \sum a_n$ converges conditionally.

(c) The series (converges absolutely, converges conditionally, diverges). (circle one)