Exam 1 Review Sheet

Monday, October 5, 2020 5:37 PM

Tip: Practice as much as you can. This will help you identify which technique you need to use

Types of Integration Techniques:

- u-sub
- Integration by parts (IBP)
- Trig
- Trig sub
- Partial Fraction Decomposition (PFD)
- Improper
- Applications of Integrals

Types of Volumes:

- Volume of a solid
- Disk method
- Washer method
- Shell method

Integration

U-sub: relationship between functions in the integral

ex. $\int x^2 \cos(x^3)\,dx$

ans. $\int x^2 \cos(x^3)\,dx$

$u = x^3$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$ replace

$\int \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u + C \rightarrow \boxed{\frac{1}{3}\sin x^3 + C}$

IBP: use when there is no relationship between nctions

To decide u: IPET — inverse, polynomial, exponential, trig.

inverse	polynomial	exponential	trig
arctan x	x	e^x	sin x
ln x	x^3	2^x	cos x

ex. $\int x\cos x\,dx$

u	dv
x +	$\cos x$
1 −	$\sin x$
0	$-\cos x$

$x\sin x - (-\cos x) + C$

$\boxed{x\sin x + \cos x + C}$

Trig: use when there are exclusively trig functions

* see end for trig stuff to memorize *

ex. $\int \sin^5 x \cos^3 x\,dx$

$$\int \sin^5 x\,(\cos^2 x)\cos x \; dx$$

$$\downarrow$$

$$\int \sin^5 x(1-\sin^2 x)\cos x\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$$\int u^5(1-u^2)\,du$$

$$\int u^5 - u^7\, du = \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$\downarrow$$

$$\boxed{\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C}$$

Trig sub: use when you see or can manipulate a function to look like (y^2-1), (y^2+1) or $(1-y^2)$

Some things to help recognize: $\sqrt[3]{\;}$, $\frac{1}{(\;)^{3/2}}$

$y^2 - 1 \rightarrow y = \sec\theta$

$y^2 + 1 \rightarrow y = \tan\theta$

$1 - y^2 \rightarrow y = \sin\theta$

y can be: x, x^2, e^x, $\frac{x+2}{2}$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

remember: complete the square

$$ax^2+bx+c+d = (ax^2 + bx + \underline{c}) + d = \left(x + \tfrac{b}{2}\right)^2 + d$$

should be positive 1 $\quad$ $c = \left(\frac{b}{2}\right)^2$

Step 1: complete the square (if necessary)

Step 2: decide the trig

Step 3: $dx \rightarrow \_\_\_\, d\theta$

Step 4: Substitute in

Step 5: trig stuff

Step 6: plug back in $\triangle$

ex. $\int \frac{1}{(x^2+4x)^{3/2}}\,dx$ $\qquad$ $\sec\theta = \frac{x+2}{2} = \frac{H}{A}$

ex. $\int \frac{1}{(x^2+4x)^{3/2}}\,dx$

$(x^2+4x+4)-4$

$(x+2)^2-4 = 4\left(\left(\frac{x+2}{2}\right)^2-1\right)$

$\sec\theta = \frac{x+2}{2} = \frac{H}{A}$

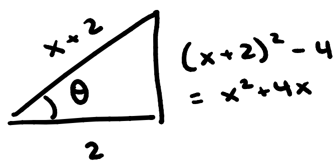
$\sec\theta\tan\theta\,d\theta = \frac{1}{2}\,dx$

$2\sec\theta\tan\theta\,d\theta = dx$

$$\int \frac{2\sec\theta\tan\theta\,d\theta}{[4(\sec^2\theta-1)]^{3/2}}$$

$$\int \frac{2\sec\theta\,\cancel{\tan\theta}\,d\theta}{8\,(\tan^{\cancel{2}}\theta)^{\cancel{3/2}}} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta = \frac{1}{4}\int \cot\theta\cdot\csc\theta\,d\theta = \frac{1}{4}\csc\theta + C$$



$$\boxed{\frac{x+2}{4(x^2+4x)} + C}$$

PFD: fraction that denominator can be broken up or seperated

$$\frac{1}{(x+a)(x+b)^2(x^2+cx+d)(x^2+ex+f)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2} + \frac{Dx+E}{x^2+cx+d} + \frac{Fx+G}{x^2+ex+f} + \frac{Hx+I}{(x^2+ex+f)^2}$$

can't be broken up

$\int \frac{Dx+E}{x^2+cx+d}\,dx$ — usub, trig sub

$\int \frac{Hx+I}{(x^2+ex+f)^2}\,dx$ — u-sub, trig sub

$\int \frac{A}{x+a}\,dx = A\ln(x+a) + C$

$\int \frac{C}{(x+b)^2} = \frac{-C}{(x+b)} + C$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

* if numerator is larger than denominator, you must do long division *

cover-up method $\frac{1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$

for $x=1$ you cover up

$$\frac{1}{(x+2)\cancel{(x-1)}} = \cancel{\frac{A}{(x+2)}} + \frac{B}{\cancel{(x-1)}}$$

$\downarrow$

plug in $x=1$ $\frac{1}{(1+2)} = B$

$x=-2$

$$\frac{1}{\cancel{(x+2)}(x-1)} = \frac{A}{\cancel{(x+2)}} + \cancel{\frac{B}{(x-1)}}$$

plug in $x=-2$ $\frac{1}{(-2-1)} = A$

ex $\displaystyle\int \frac{x^4+3x^2+2x}{(x^2-9)}\,dx$

$$\begin{array}{r|l} & \quad\quad\quad x^2+12 \\ x^2-9 & x^4+0x^3+3x^2+2x \\ & x^4 \quad\quad -9x^2 \\ \hline & \quad\quad 12x^2+2x \\ & \quad\quad 12x^2 \quad -108 \\ \hline & \quad\quad\quad 2x+108 \end{array}$$

$\Longrightarrow$ $\displaystyle\int x^2+12+\frac{2x+108}{(x+3)(x-3)}\,dx$

$$\frac{2x+108}{\cancel{(x+3)}(x-3)} = \frac{A}{\cancel{x+3}} + \cancel{\frac{B}{x-3}}$$

$x=3$

$\frac{2(3)+108}{3+3} = B$

$B=19$

$x=-3$

$\frac{2(-3)+108}{-3-3} = A$

$A=-17$

$$\int x^2+12+\frac{-17}{x+3}+\frac{19}{x-3}\,dx = \boxed{\frac{x^3}{3}+12x-17\ln(x+3)+19(x-3)+C}$$

Improper: when the bounds do not plug into the solution to give a numerical answer, specifically ∞ or $-\infty$

ex $\displaystyle\int_1^\infty x^3 e^{-x^4}\,dx$

start by ignoring the bounds and solve as usual

$u=-x^4$

$du=-4x^3dx$

$-\frac{1}{4}\int e^u\,du$

$$\lim_{b\to\infty} -\frac{1}{4}e^{-x^4}\Big|_1^b = \lim_{b\to\infty} \cancelto{0}{\frac{-1}{4e^{x^4}}} - \left(-\frac{1}{4}e^{-1}\right) = \boxed{\frac{1}{4e}}$$

$$\lim_{b\to\infty} -\frac{1}{4}e^{-x^4}\Big|_1^b = \cancel{\lim_{b\to\infty} -\frac{1}{4e^{x^4}}} - \left(-\frac{1}{4}e^{-1}\right) = \boxed{\frac{1}{4e}}$$

plug in bounds

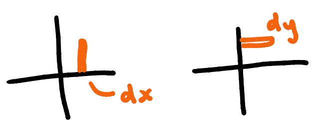
Applications of Integrals

$$\underbrace{W}_{\text{work}} = \int \underbrace{F}_{\text{force}}\,dx \qquad \qquad F_g = \underbrace{m}_{\text{mass}}g \quad \text{(force of gravity)}$$

$$\underbrace{F}_{\text{Force}} = \int \underbrace{P}_{\text{pressure}}\,dA \qquad \qquad P = \underbrace{\rho}_{\text{density}} g h$$

Volumes

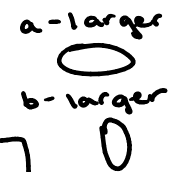
the integral you take (dx, dy) is determined by which axis the cut is ⊥ to



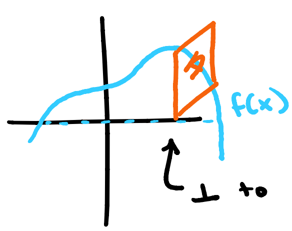
circles $(x-h)^2 + (y-k)^2 = r^2$

have same coefficient

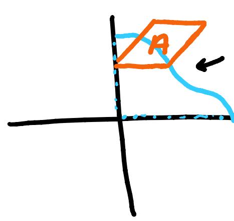
ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$



Volume of a solid



$$\int_0^{f(x)} A(x)\,dx$$

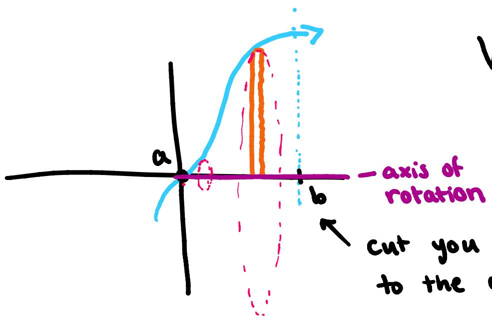


$$\int_0^{f(y)} A(y)\,dy$$

common areas: squares, right isoselese △, equilateral triangle, semicircle

s^2, $\frac{1}{2}s^2$, $\frac{\sqrt{3}}{4}s^2$, $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2$

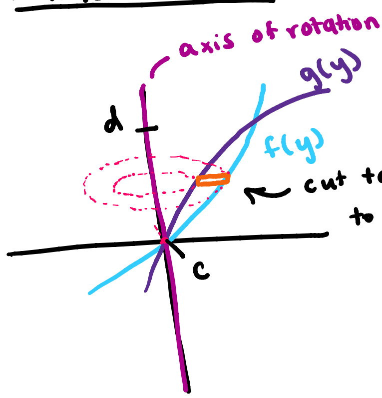
Disk Method



$$V = \pi \int_a^b (f(x))^2 \, dx$$

cut you take is $\perp$
to the axis of rotation

Washer Method



cut taken is $\perp$
to axis of rotatio

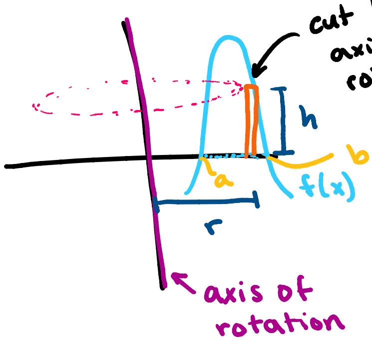
outer
radius

$$V = \pi \int_a^b \text{OR}^2 - \text{IR}^2 \, dy$$

inner
radius

$$V = \pi \int_c^d f(y)^2 - g(y)^2 \, dy$$

Shell Method



cut ∥ to
axis of
rotation

$$V = 2\pi \int_a^b (\text{radius})(\text{height}) \, dx$$

$$\Downarrow$$

$$2\pi \int_a^b x \;\; f(x) \, dx$$

if axis of rotation is not the y axis,
the radius will be $(x \pm b)$ when b
is the axis of rotation

if axis of rotation
is $x = -1$, radius $= (x+1)$

Trig Memorization:

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

SOH CAH TOH

$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$

$\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$

$\tan\theta = \frac{\text{opp.}}{\text{adj.}}$

$\frac{d}{d\theta}\sin\theta = \cos\theta$

$\frac{d}{d\theta}\cos\theta = -\sin\theta$

$\frac{d}{d\theta}\tan\theta = \sec^2\theta$

$\frac{d}{d\theta}\csc\theta = -\cot\theta\csc\theta$

$\frac{d}{d\theta}\sec\theta = \tan\theta\sec\theta$

$\frac{d}{d\theta}\cot\theta = -\csc^2\theta$

$\int \tan\theta \, d\theta \equiv -\ln|\cos x| + c$

$\int \cot\theta \, d\theta = \ln|\sin x| + c$

$\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + c$

$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$

$\int \sec^3\theta \, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + c$