Multiple Choice Answers:

E1A: a a e b c, d b c b c

E1B: b e c b c, b b b d e

E1C: e b c a d, c d b b a

E1D: d b c a b, a b a b e

Part I: Multiple Choice

1. To evaluate the integral $\int (2x+1)\tan^{-1}(\sqrt{x})\,dx$ using integration by parts, we should choose:
   (a) $u = \tan^{-1}\sqrt{x}$ and $v' = 2x+1$
   (b) $u = 2x+1$ and $v' = \tan^{-1}\sqrt{x}$
   (c) $u = \sqrt{x}$ and $v' = \tan^{-1}\sqrt{x}$
   (d) $u = \tan^{-1} x$ and $v' = 2x+1$
   (e) $u = \sqrt{x}$ and $v' = 2x+1$

   (Ans. Choose $u : IPET$, we let $u = \arctan(\sqrt{x}),\ dv = (2x+1)\ dx$)

2. The partial fraction decomposition of $\dfrac{-x+7}{x(x+7)(x-1)^2(x+1)}$ has the form $\dfrac{A}{x} + \dfrac{B}{x+7} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} + \dfrac{E}{x+1}$. Find $A - D$.
   (a) $\dfrac{5}{8}$ (b) $-\dfrac{7}{6}$ (c) $\dfrac{11}{8}$ (d) $\dfrac{13}{6}$ (e) $\dfrac{3}{8}$

   (Ans. Using '**Cover-up method**, A and D can be found easily and quickly: $A = \dfrac{7}{7\cdot 1\cdot 1} = 1$ and $D = \dfrac{-1+7}{1\cdot 8\cdot 2} = \dfrac{3}{8} \longrightarrow A - D = 1 - \dfrac{3}{8} = \dfrac{5}{8}$.)

3. Evaluate the definite integral:

$$\int_0^{\pi/2} \sin^3(x)\cos^3(x)\,dx$$

   (a) $-\dfrac{1}{12}$ (b) $-\dfrac{1}{16}$ (c) 0 (d) $\dfrac{1}{16}$ (e) $\dfrac{1}{12}$

   (Ans. Odd powers of sine or cosine are welcome news. Since both functions have odd powers here, we can save either one to be the 'du in the $u-$sub. Let's save one $\cos x$ and let $u = \sin x$ to set up for the $u-$sub: $\displaystyle\int_0^{\pi/2} u^3(1-u^2)\,du = \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6}\Big|_0^{\pi/2} = \frac{1}{12}$).

4. Evaluate the definite integral

$$\int_2^{\infty} \frac{1}{x^3 - x^2}\, dx$$

(a) $\ln\left(\frac{1}{2}\right)$ (b) $\ln(2) - \frac{1}{2}$ (c) $-2 - \ln\left(\frac{1}{2}\right)$ (d) $\frac{1}{2}$ (e) This integral diverges

(Ans. Perform a PFD and using cover-up method, we find B and C easily and quickly: $\frac{1}{x^3 - x^2} = \frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \longrightarrow B = -1, C = 1$. Plug in say, $x = 2$ to both sides of the equation, we obtain $A = -1$.

Hence $\int_2^{\infty} \frac{1}{x^3 - x^2}\, dx = \lim_{t\to\infty} \int_2^t -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\, dx = \lim_{t\to\infty} -\ln|x| + \frac{1}{x} + \ln|x-1| = \lim_{t\to\infty} \frac{1}{x} + \ln\left(\frac{x-1}{x}\right)\Big|_2^t =$ $0 + 0 - (\frac{1}{2} + \ln\left(\frac{2-1}{2}\right)) = -(\frac{1}{2} - \ln 2) = \ln 2 - \frac{1}{2}$)

5. When you calculate $\int \frac{8x^2}{x^4 - 16}\, dx$, which of the following appears as a term of the solution?

(a) $-\frac{3}{x+2}$ (b) $-\frac{32}{3(x-2)^3}$ (c) $2\arctan\left(\frac{x}{2}\right)$ (d) $\ln|x^2 + 4|$

(Ans. PFD: $\frac{8x^2}{x^4 - 16} = \frac{8x^2}{(x^2+4)(x+2)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2} + \frac{D}{x-2}$.

NOTE: the integral of the first term produces a $\ln(x^2+4)$ and an $\arctan\left(\frac{x}{2}\right)$, the second and third terms produce a $\ln|x+2|$ and $\ln|x-2|$ receptively, so we can eliminate the first 2 answer choices.

Cover-Up method gives $D = 1$, $C = -1$ and plug in $x = 0$ to both sides to obtain a $B = 4$. Plug in say, $x = 1$ to both sides to obtain $A = 0$. Hence,

$$\int \frac{Ax+B}{x^2+4} = \frac{4}{x^2+4} = \frac{4}{2}\arctan\left(\frac{x}{2}\right))$$

6. Evaluate the following indefinite integral:

$$\int \frac{1}{(4x - x^2)^{3/2}}\, dx$$

(a) $\dfrac{2}{\sqrt{4x - x^2}} + C$

(b) $\dfrac{2}{5}(4x - x^2)^{5/2} + C$

(c) $\dfrac{8 - 4x}{\sqrt{4x - x^2}} + C$

(d) $\dfrac{x - 2}{4\sqrt{4x - x^2}} + C$

(Ans. Complete the square first, then do a trig-sub: $(4x - x^2) = 4 - (x-2)^2 \longrightarrow (x-2) = 2\sin\theta,\ dx = 2\cos\theta\ d\theta$ and $(4x - x^2)^{3/2} = (\sqrt{4 - (x-2)^2})^3 = 2^3\cos^3\theta$. $\displaystyle\int \frac{1}{(4x - x^2)^{3/2}}\, dx = \int \frac{2\cos\theta}{2^3\cos^3\theta}\, d\theta = \frac{1}{4}\int \sec^2\theta\ d\theta = \frac{1}{4}\tan\theta + C$. Use the triangle to trace back, $\tan\theta = \dfrac{x-2}{\sqrt{4 - (x-2)^2}}$, We obtain $\dfrac{1}{4}\tan\theta + C = \dfrac{1}{4}\dfrac{x-2}{\sqrt{4x - x^2}} + C$)

7. Evaluate the following indefinite integral:

$$\int x^2 e^{x/3}\, dx$$

(a) $x^3 e^{x/3} + C$

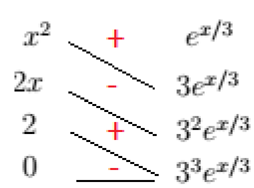
(b) $3x^2 e^{x/3} - 18x e^{x/3} + 54 e^{x/3} + C$

(c) $3x^2 e^{x/3} - 18x e^{x/3} - 9 e^{x/3} + C$

(d) $\dfrac{1}{3}x^2 - \dfrac{2}{9}x e^{x/3} + \dfrac{1}{27}e^{x/3} + C$

(Ans. Use the tabular method (IBP), we obtain the answer quickly: $u = x^3,\ dv = e^{x/3}$,

$$\int x^2 e^{x/3}\, dx = 3x^2 e^{x/3} - 18x e^{x/3} + 54 e^{x/3} + C\ )$$



8. Consider $\dfrac{x^4+10x^2}{(x^2+1)(x^2+9)}$. How many of the following statments are **true**?

- The largest coefficient that appears when you do the partial fraction decomposition is $\dfrac{9}{8}$.
- $\dfrac{x^4+10x^2}{(x^2+1)(x^2+9)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+9}$ for some constants A, B, C and D.
- In the $\dfrac{Ax+B}{x^2+1}$ term, $A = 0$.

(a) 0 (b) 1 (c) 2 (d) 3

(Ans. First, use Long division to get a proper integrand: $\dfrac{x^4+10x^2}{(x^2+10x^2+9)} = 1 + \dfrac{-9}{(x^2+1)(x^2+9)}$
Perform PFD, we get

$$\frac{-9}{(x^2+1)(x^2+9)} = \frac{0x - \frac{9}{8}}{x^2+1} + \frac{0x + \frac{9}{8}}{x^2+9})$$

9. Evaluate the indefinite integral:

$$\int \sec(x)\tan^2(x)\, dx$$

(a) $\dfrac{1}{2}\Big(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\Big) + C$
(b) $\dfrac{1}{2}\Big(\sec(x)\tan(x) - \ln|\sec(x)+\tan(x)|\Big) + C$
(c) $\dfrac{1}{2}\sec(x)\tan(x) + \dfrac{1}{2}\ln|\tan(x)| + C$
(d) $\sec(x)\tan(x) - \sec^3(x) + C$

(Ans. Let $\tan^2 x = \sec^2 x - 1$, we have $\int \sec(x)\tan^2(x)\, dx = \int \sec^3 x - \sec x\, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| = \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x| + C)$)

10. Evaluate the following definite integral:

$$\int_0^2 \frac{x}{x^4+1}\, dx$$

(a) $\dfrac{\arctan(2)}{2}$ (b) $\dfrac{\ln(17)}{4}$ (c) $\dfrac{\arctan(4)}{2}$ (d) $\dfrac{\ln(17)}{2}$

(Ans.Through a simple u-sub $u = x^2, \frac{1}{2}dx = du$, the integral is transformed to an arctangent: $\int_0^2 \frac{x}{x^4+1}\, dx = \frac{1}{2}\int \frac{1}{u^2+1}\, du = \frac{1}{2}\arctan(x^2)\Big|_0^2 = \frac{\arctan(4)}{2}$)

Question	Form A	Form B	Form C	Form D
1	**C**	**B**	**B**	**C**
2	**E**	**C**	**C**	**B**
3	**A**	**D**	**E**	**E**
4	**D**	**E**	**D**	**A**

Name: ______________________________

Part II: Free Response

FR Scores	
1	/5
2	/4
3	/6
4	/5
FR Total	/20

1. (a) Evaluate the indefinite integral:

$$\int e^{-2x} \sin x \, dx$$

(Ans. Use IBP where $u = e^{-2x}$, $dv = \sin x \, dx$, we have

e^{-2x}	+	$\sin x$
$-2e^{-2x}$	-	$-\cos x$
$(-2)^2 e^{-2x}$	+	$-\sin x$

$$\int e^{-2x} \sin(x) \, dx = -e^{-2x} \cos x - 2e^{-2x} \sin x - 4 \int e^{-2x} \sin x \, dx$$

$$5 \int e^{-2x} \sin(x) \, dx = -e^{-2x} \cos x - 2e^{-2x} \sin x + C$$

$$\int e^{-2x} \sin(x) \, dx = \frac{1}{5}(-e^{-2x} \cos x - 2e^{-2x} \sin x) + C$$

)

$$\int e^{-2x} \sin(x) \, dx = \underline{-\frac{1}{5} e^{-2x} (\cos x + 2 \sin x) + C}$$

(b) Calculate $\int_0^{\infty} e^{-2x} \sin x \, dx$

(Ans. $\int_0^{\infty} e^{-2x} \sin x \, dx = -\frac{1}{5} \lim_{t \to \infty} \left(\frac{\cos t + 2 \sin t}{e^{2t}} - \frac{\cos 0 + 2 \sin 0}{e^0} \right) = \frac{1}{5}$)

2. Use this list of integrals to answer the questions below.

i) $\int x\cos(x^2)\,dx$	*ii)* $\int xe^{x^2}\,dx$	*iii)* $\int \ln(x)\,dx$
iv) $\int x\cos(x)\,dx$	*v)* $\int e^{x^2}\,dx$	*vi)* $\int \cos(x^2)\,dx$
vii) $\int x^2e^x\,dx$	*viii)* $\int e^x\sin(e^x)\,dx$	*ix)* $\int \frac{1}{(1-x^2)^{3/2}}\,dx$

(a) Which of the above integrals can be done using **only** integration by parts? You do not need to justify your answers.

(Ans. IBP: (iii), (iv), (vii))

(b) Which of the integrals above can be done using **only** a u-substitution and/or trigonometric substitution? You do not need to justify your answers.

(Ans. IBP: (i), (ii), (viii), (ix))

3. Evaluate the integral:

$$\int \arctan\left(\frac{1}{x-7}\right)\, dx$$

(Ans. IBP, let $u = \arctan\left(\frac{1}{x-7}\right)$, $dv = dx$)

$\arctan\left(\frac{1}{x-7}\right)$ $+$ dx

$\frac{1}{1+\left(\frac{1}{x-7}\right)^2} \cdot \left(\frac{-1}{(x-7)^2}\right)$ $-$ x

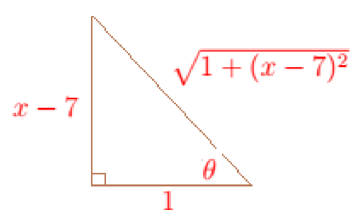
$= \left(\frac{-1}{(x-7)^2+1}\right)$

$$\int \arctan\left(\frac{1}{x-7}\right)\, dx = x\arctan\left(\frac{1}{x-7}\right) + \int \frac{x}{1+(x-7)^2}\, dx$$

trig-sub: $x - 7 = \tan\theta \qquad dx = \sec^2\theta\, d\theta$

$$\int \frac{x}{1+(x-7)^2}\, dx = \int \frac{(\tan\theta + 7)\cdot \sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \ln|\sec\theta| + 7\theta$$



$$= x\arctan\left(\frac{1}{x-7}\right) + \ln|\sqrt{1+(x-7)^2}| + 7\arctan(x-7) + C$$

$$= x\arctan\left(\frac{1}{x-7}\right) + \frac{1}{2}\ln(x^2+14x+50) + 7\arctan(x-7) + C$$

Answer, Option (II)−

(One can also start with a u−sub, $u = \dfrac{1}{x-7}$, $x - 7 = \dfrac{1}{u}$, $dx = -\dfrac{1}{u^2}$. Then,

$$\int \arctan\left(\frac{1}{x-7}\right)\,dx = \int \frac{-\arctan u}{u^2}\,du. \text{ Use IBP, we have}$$
$$= \frac{1}{u}\arctan u - \int \frac{1}{u(1+u^2)}\,du.$$

Use PFD to find the second integral $\displaystyle\int \frac{1}{u(1+u^2)}\,du = \int \frac{1}{u} - \frac{u}{1+u^2}\,du$

We have $\displaystyle\int \arctan\left(\frac{1}{x-7}\right)\,dx = \frac{1}{\frac{1}{x-7}}\arctan\left(\frac{1}{x-7}\right) - \ln|\frac{1}{x-7}| + \frac{1}{2}\ln|1 + (\frac{1}{x-7})^2|)$

$$= (x-7)\arctan\left(\frac{1}{x-7}\right) + \ln|x-7| + \frac{1}{2}\ln\left|\frac{(x-7)^2+1}{(x-7)^2}\right|$$
$$= (x-7)\arctan\left(\frac{1}{x-7}\right) + \ln|x-7| + \frac{1}{2}(\ln|x^2-14x+50| - \ln((x-7)^2))$$
$$= (x-7)\arctan\left(\frac{1}{x-7}\right) + \frac{1}{2}(\ln(x^2-14x+50) + C.\ )$$

4. Evaluate the integral $\int \frac{8x+7}{x^2+2x+2}\,dx$

(Ans. Note the denominator is irreducible, use the 2-step process to set up the integral efficiently:

(1) $U-$-sub:

$u = x^2+2x+2$

$du = (2x+2)\,dx$

$4\,du = (8x+8)\,dx$

(2) Comp.Sqr:

$x^2+2x+2 = (x+1)^2+1$

$$\int \frac{8x+7}{x^2+2x+2}\,dx = \int \frac{8x+8}{x^2+2x+2}\,dx - \int \frac{1}{(x+1)^2+1}\,dx = \left(4\int \frac{1}{u}\,du\right) - \arctan(x+1) + C$$

$$= 4\ln(x^2+2x+2) - \arctan(x+1) + C.\ )$$

$\int \frac{8x+7}{x^2+2x+2}\,dx =$ ______________________________

1a) $\int e^{-2x}\sin x\,dx$ IBP

$\frac{d}{dx}$		$\int dx$
e^{-2x}	+	$\sin x$
$-2e^{-2x}$	−	$-\cos x$
$4e^{-2x}$	+	$-\sin x$

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x\,dx$$

$$5\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x + C$$

$$\boxed{\int e^{-2x}\sin x\,dx = \frac{-e^{-2x}\cos x - 2e^{-2x}\sin x}{5} + C}$$

1b) $$\int_0^\infty e^{-2x}\sin x\,dx = \lim_{r\to\infty}\int_0^r e^{-2x}\sin x\,dx = \lim_{r\to\infty} -\frac{e^{-2x}\cos x - 2e^{-2x}\sin x}{5}\Big|_0^r$$

$$= \lim_{r\to\infty}\frac{-e^{-2r}(\cos r + 2\sin r)}{5} + 1/5 = 0 + 1/5 = \boxed{1/5}$$

2a) iv, vii, iii | 2b) i, ii, vi, viii, ix

3) $\int \arctan\left(\frac{1}{x-7}\right)dx$ IBP: $u = \arctan\left(\frac{1}{x-7}\right)$ $dv = dx$

$du = \frac{1}{1+\left(\frac{1}{x-7}\right)^2} \cdot \frac{-1}{(x-7)^2}dx$ $v = x$

$-\frac{1}{(x-7)^2+1}$

$= x\arctan\left(\frac{1}{x-7}\right) + \int \frac{x\,dx}{(x-7)^2+1}$

$= x\arctan\left(\frac{1}{x-7}\right) + \int \frac{x-7}{(x-7)^2+1}dx + \int \frac{7}{(x-7)^2+1}dx$

arctan integral

$u = (x-7)^2+1$

$du = 2(x-7)dx$

$\frac{1}{2}du = (x-7)dx$

$\frac{1}{2}\int du/u$

$= x\arctan\left(\frac{1}{x-7}\right) + \frac{1}{2}\ln|(x-7)^2+1| + 7\arctan(x-7) + C$

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4) $\int \frac{8x+7}{x^2+2x+2}dx$ Complete the □: x^2+2x+2

$x^2+2x+1-1+2$

$(x+1)^2+1$

$= \int \frac{8x+7}{(x+1)^2+1}dx$

$= \int \frac{8x+8}{(x+1)^2+1}dx - \int \frac{1}{(x+1)^2+1}dx = \int \frac{4du}{u} - \int \frac{1}{(x+1)^2+1}dx$

$u = (x+1)^2+1$

$du = 2(x+1)\,dx$

$4du = 8x+8\,dx$

$= 4\ln|u| - \arctan(x+1) + C$

$= 4\ln|(x+1)^2+1| - \arctan(x+1) + C$