Exam 2 Key

Part I:	
Question	Answer
1	D
2	E
3	D
4	B
5	A
6	D
7	B
8	B
9	B
10	C
11	C
12	E
13	D
14	C

Calculus I: MAC2311 **Name:** ________________________

Exam 2

<u>Part</u> <u>II</u> <u>Instructions</u>: 5 free response questions

For Instructor Use Only:

FR 1	
FR 2	
FR 3	
FR 4	
FR 5	
Total Points	

1. Complete both parts of the problem concerning the function $f(x) = 2x - 4\cos(x)$.

   (*a*) How many tangent lines to the function $f(x)$ are horizontal in the interval $[0, 2\pi]$?

   Solution
   $f'(x) = 2 + 4\sin x = 0$, so $\sin x = -\frac{1}{2}$. This equation has two solutions on the interval $[0, 2\pi]$: $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. That is, $f(x)$ has two horizontal tangent lines in the interval $[0, 2\pi]$.

   (*b*) Write down the equations of these horizontal tangent lines.

   Solution

$$f\left(\frac{7\pi}{6}\right) = \frac{7\pi}{3} - 4\cos\left(\frac{7\pi}{6}\right) = \frac{7\pi}{3} + 2\sqrt{3}$$

$$f\left(\frac{11\pi}{6}\right) = \frac{11\pi}{3} - 4\cos\left(\frac{11\pi}{6}\right) = \frac{11\pi}{3} - 2\sqrt{3}$$

   The equations of the horizontal lines are $y = \frac{7\pi}{3} + 2\sqrt{3}$ and $y = \frac{11\pi}{3} - 2\sqrt{3}$.

2. Find an equation for the line tangent to the graph $y = \dfrac{(x^2+3x+1)e^x}{\cos(x)}$ at $x = 0$.

Solution

$$\begin{aligned}
y' &= \frac{\cos x \cdot \frac{d}{dx}[(x^2+3x+1)e^x] - (x^2+3x+1)e^x \cdot \frac{d}{dx}[\cos x]}{(\cos x)^2} \\
&= \frac{\cos x \cdot \left[(x^2+3x+1) \cdot \frac{d}{dx}[e^x] + e^x \cdot \frac{d}{dx}[x^2+3x+1]\right] - (x^2+3x+1)e^x \cdot - \sin x}{\cos^2 x} \\
&= \frac{\cos x \cdot \left[(x^2+3x+1)e^x + e^x(2x+3)\right] + (x^2+3x+1)e^x \sin x}{\cos^2 x}
\end{aligned}$$

Plugging in $x = 0$,

$$y'(0) = \frac{1 \cdot [1 \cdot 1 + 1 \cdot 3] + 1 \cdot 1 \cdot 0}{1^2} = 4.$$

When $x = 0$, $y(0) = 1$. The equation of the tangent line is therefore $y = 4x + 1$.

3. Find all points on the curve $x^2 - xy + y^2 = 3$ at which there is a vertical tangent line. Write your answer(s) in the form (x, y) for each coordinate pair.

Solution
Differentiating both sides with respect to x,

$$\begin{aligned}
\frac{d}{dx}\left[x^2 - xy + y^2\right] &= \frac{d}{dx}[3] \\
2x - xy' - y + 2yy' &= 0 \\
(-x + 2y)y' &= -2x + y \\
y' &= \frac{-2x + y}{-x + 2y}
\end{aligned}$$

Vertical tangent lines occur when $-x + 2y = 0$, so $x = 2y$. Plugging into the curve,

$$\begin{aligned}
(2y)^2 - 2y \cdot y + y^2 &= 3 \\
3y^2 &= 3 \\
y &= \pm 1
\end{aligned}$$

When $y = 1$, $x = 2$. When $y = -1$, $x = -2$. The points on the curve where there is a vertical tangent are $(2, 1)$ and $(-2, -1)$.

4. Evaluate the following:

- $\frac{d}{dx}\left(\sin^{-1}(x)\right),\ \left(\frac{d}{dx}(\arcsin(x))\right)$

  Solution

  $$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

- $\frac{d}{dx}\left(\csc^{-1}(x^2)\right),\ \left(\frac{d}{dx}(\operatorname{arccsc}(x^2))\right)$

  Solution

  $$\frac{d}{dx}\left[\csc^{-1}(x^2)\right] = \frac{-1}{x^2\sqrt{(x^2)^2-1}}\cdot\frac{d}{dx}\left[x^2\right] = \frac{-2x}{x^2\sqrt{x^4-1}} = \frac{-2}{x\sqrt{x^4-1}}$$

- $\frac{d}{dx}\left(\tan^{-1}(3x)\right),\ \left(\frac{d}{dx}(\arctan(3x))\right)$

  Solution

  $$\frac{d}{dx}\left[\tan^{-1}(3x)\right] = \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}[3x] = \frac{3}{1+9x^2}$$

- $\frac{d}{dx}\left(\cos^{-1}(2x+1)\right),\ \left(\frac{d}{dx}(\arccos(2x+1))\right)$

  Solution

  $$\frac{d}{dx}\left[\cos^{-1}(2x+1)\right] = \frac{-1}{\sqrt{1-(2x+1)^2}}\cdot\frac{d}{dx}[2x+1] = \frac{-2}{\sqrt{1-(2x+1)^2}}$$

5. A stone is thrown upward from a 60 meter tall cliff so that its height above the ground is $h(t) = 60 + 4t - t^2$ for $t \geq 0$.

(*a*) When does the stone reach its highest point?

Solution
$v(t) = h'(t) = 4 - 2t = 0$, so $t = 2$. The stone reaches its highest point after 2 seconds.

(*b*) When does the stone hit the ground?

Solution

$$\begin{aligned} h(t) &= 0 \\ 60 + 4t - t^2 &= 0 \\ t^2 - 4t - 60 &= 0 \\ (t-10)(t+6) &= 0 \\ t &= 10 \end{aligned}$$

The stone hits the ground after 10 seconds. Notice that $t = -6$ is not a solution because time starts when $t = 0$

(*c*) What is the total vertical distance traveled by the stone from when it is thrown to when it hits the ground?

Solution

$$|h(2) - h(0)| + |h(10) - h(2)| = |64 - 60| + |0 - 64| = 4 + 64 = 68 \text{ meters}$$

Part III:

MC Key

1	B
2	D
3	A
7	A
13	A
14	C

Calculus I: MAC2311 **Name:** ____________________

Part IV Instructions: 2 free response questions

For Instructor Use Only:

FR 1	
FR 2	
Total Points	

1pt for taking ln of both sides

1. (7 pts) Use logarithmic differentiation to find the derivative of $f(x) = \sqrt{\frac{x^2\cos^3(x)}{e^x\sqrt{x}}}$.

$$\ln(f(x)) = \ln\left(\sqrt{\frac{x^2\cos^3(x)}{e^x\sqrt{x}}}\right)$$

1 pt for using log laws to rewrite

$$\ln(f(x)) = \frac{1}{2}\left(2\ln x + 3\ln(\cos(x)) - x - \frac{1}{2}\ln(x)\right)$$

$$\ln(f(x)) = \ln x + \frac{3}{2}\ln(\cos(x)) - \frac{x}{2} - \frac{1}{4}\ln(x)$$

1pt for correctly differentiating the left hand side

$$\frac{f'(x)}{f(x)} = \boxed{\frac{1}{x}} + \boxed{\frac{3}{2}\cdot\frac{1}{\cos(x)}(-\sin(x))} - \boxed{\frac{1}{2}} - \boxed{\frac{1}{4x}}$$

$\frac{1}{2}$ pt for each correctly differentiated term on right hand side (2 pts total)

$$f'(x) = f(x)\left(\frac{1}{x} - \frac{3\sin(x)}{2\cos(x)} - \frac{1}{2} - \frac{1}{4x}\right)$$

1 pt multiplying by f(x)

not required

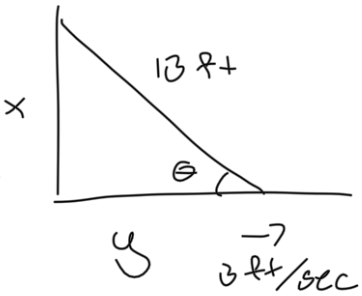
$$= \sqrt{\frac{x^2\cos^3(x)}{e^x\sqrt{x}}}\left(\frac{1}{x} - \frac{3}{2}\tan(x) - \frac{1}{2} - \frac{1}{4x}\right)$$

1 pt for substituting the function back in

2. (7 pts) Suppose a 13 foot ladder rests against a wall. If the bottom of the ladder slides away from the wall at a rate of 3 feet per second, at what rate does the angle the ladder makes with the ground change when the top of the ladder is 5 feet from the ground?

$$\frac{dy}{dt} = 3 \text{ ft/sec}$$

want $\frac{d\theta}{dt}$ when $x = 5$ ft

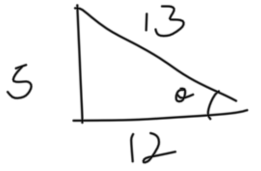


1 pt for clearly labeling variables

$$\cos\theta = \frac{y}{13}$$

← 1 pt for the equation

$$\boxed{-\sin\theta}\boxed{\frac{d\theta}{dt}} = \boxed{\frac{1}{13}}\boxed{\frac{dy}{dt}}$$



$$-\boxed{\left(\frac{5}{13}\right)}\left(\frac{d\theta}{dt}\right) = \frac{1}{13}\boxed{(3)}$$

$\frac{1}{2}$ pt for each circled item

2 pts total for correct derivative

1 pt total for plugging in the correct values

$$\frac{d\theta}{dt} = -\frac{3}{5} \text{ rad/sec}$$

1 pt answer

1 pt units