Exam 3 Review Answer Key

Disclaimer: This answer key should not be provided to students.

1. Solve the differential equation.

   (a) $y' = xy^2$ $\left(\text{Answer: } y = -\dfrac{2}{x^2 + C}\right)$

   (b) $y' = \dfrac{y \ln x}{x}$ $\left(\text{Answer: } y = Ce^{[(\ln(x))^2/2]}\right)$

   (c) $y' = 2 - y$ with initial condition $y(0) = 3$. $\left(\text{Answer: } y = 2 + e^{-x}\right)$

   (d) $y' = 3xy - 2x$ with initial condition $y(0) = 1$. $\left(\text{Answer: } y = \dfrac{1}{3}e^{[3x^2/2]} + \dfrac{2}{3}\right)$

2. Use Euler's method to approximate the indicated function value to 3 decimal places, using $h = 0.1$.

   (a) $\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 2$; find $y(0.5)$. (Answer: $y(0.5) \approx 8.273$)

   (b) $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$, $y(1) = 0$; find $y(1.4)$. (Answer: $y(1.4) \approx 0.452$)

3. Find the value of k that will make f a probability density function on the indicated domain.

   (a) $f(x) = k\sqrt{x}$; $1 \le x \le 4$ $\left(\text{Answer: } k = \dfrac{3}{14}\right)$

   (b) $f(x, y) = ky\sqrt{x}$; $D = \{0 \le x \le 2;\ 1 \le y \le 2\}$ $\left(\text{Answer: } k = \dfrac{1}{2\sqrt{2}}\right)$

4. f is a probability density function for a random variable X defined on the given interval. Find the indicated probabilities.

   (a) $f(x) = \dfrac{1}{4\sqrt{x}}$; $[1, 9]$

      (i) $P(X \ge 4) = \dfrac{1}{2}$

      (ii) $P(X \le 4) = \dfrac{1}{2}$

      (iii) $P(1 \le X < 8) = \sqrt{2} - \dfrac{1}{2}$

      (iv) $P(X = 3) = 0$

   (b) $f(x) = 2xe^{-x^2}$; $[0, \infty)$

      (i) $P(X \le 4) = 1 - e^{-16}$

      (ii) $P(X \ge 2) = e^{-4}$

      (iii) $P(1 < X < 2) = \dfrac{e^3 - 1}{e^4}$

5. f is a joint probability density function for the random variables X and Y on D. Find the indicated probabilities.

   (a) $f(x,y) = xy; \quad D = \{0 \le x \le 1;\ 0 \le y \le 2\}$

      (i) $P(0 \le X \le 1;\ 0 \le Y \le 1) = \dfrac{1}{4}$

      (ii) $P(X + 2Y \le 1) = \dfrac{1}{96}$

   (b) $f(x,y) = \dfrac{1}{12}(x+y); \quad D = \{0 \le x \le 2;\ 1 \le y \le 3\}$

      (i) $P(1 \le X \le 2;\ 0 \le Y \le 1) = 0$

      (ii) $P(Y \ge 1;\ X + Y \le 3) = \dfrac{7}{18}$

6. Find the mean, variance, and standard deviation of the random variable X associated with the probability density function over the indicated interval.

   (a) $f(x) = \dfrac{3}{8}x^2 \quad [0,2]$

      Answer: $\mu = \dfrac{3}{2}, \quad \sigma^2 = \dfrac{3}{20}, \quad \sigma = \sqrt{\dfrac{3}{20}}$

   (b) $f(x) = \dfrac{3}{x^4} \quad [1,\infty)$

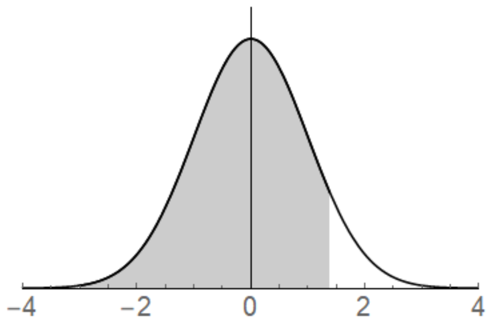
      Answer: $\mu = \dfrac{3}{2}, \quad \sigma^2 = \dfrac{3}{4}, \quad \sigma = \dfrac{\sqrt{3}}{2}$

   (c) $f(x) = \dfrac{5}{2}x^{3/2} \quad [0,1]$

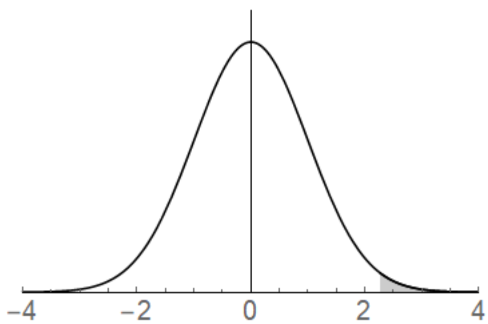
      Answer: $\mu = \dfrac{5}{7}, \quad \sigma^2 = \dfrac{20}{441}, \quad \sigma = \dfrac{\sqrt{20}}{21}$

7. Let Z be the standard normal random variable. Find the value of the given probability, then make a sketch of the area under the standard normal curve associated with this probability.

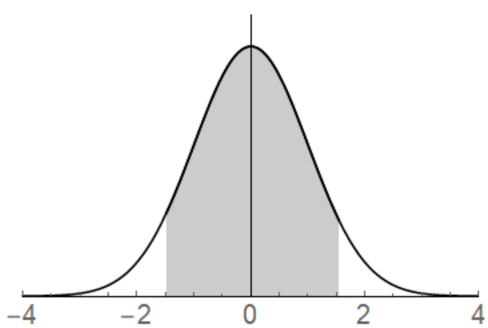
(a) $P(Z < 1.38) = 0.9162$



(b) $P(Z > 2.27) = 0.0116$



(c) $P(-1.48 < Z < 1.54) = 0.8688$



8. The medical records of infants delivered at Shands Hospital show that the infants' lengths at birth (in inches) are normally distributed with a mean of 20 in and a standard deviation of 2.6 in. Find the probability that an infant selected at random from among those at the hospital measures:

   (a) More than 22 in.

   Answer: $P(Z > 0.77) = 0.2206$

   (b) Less than 18 in.

   Answer: $P(Z < -0.77) = 0.2206$

   (c) Between 19 and 21 in.

   Answer: $P(-0.38 < Z < 0.38) = 0.296$

9. Let Z be the standard normal random variable. Find the values of z such that:

   (a) $P(Z < z) = 0.8907$ (Answer: $z = 1.23$)

   (b) $P(Z > z) = 0.9678$ (Answer: $z = -1.85$)

10. Find the third degree Taylor polynomial of the function $f(x)$ centered at $x = a$.

    (a) $f(x) = \ln(x+1); \quad a = 0$

    Answer: $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

    (b) $f(x) = \sqrt{x}; \quad a = 4$

    Answer: $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

    (c) $f(x) = \frac{1}{1-x}; \quad a = 2$

    Answer: $P_3(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3$

11. Using your answer from 10(a), estimate

$$\int_0^{1/2} \ln(x+1)dx.$$

Compare your result with the exact value of the integral using integration by parts. What is the exact error associated with the approximation?

Answer: $\int_0^{1/2} P_3(x)\,dx = \int_0^{1/2} x - \frac{1}{2}x^2 + \frac{1}{3}x^3\,dx = \frac{7}{64} = 0.109375$

$\int_0^{1/2} \ln(x+1)\,dx = \frac{3}{2}\ln\left(\frac{3}{2}\right) - \frac{1}{2} \approx 0.1081977$

Exact error$=0.109375 - 0.1081977 = 0.0011773$

12. Approximate $\sqrt{1.1}$ using the second degree Taylor polynomial of $f(x) = \sqrt{x+1}$ centered at $x = 0$. Without calculating $\sqrt{1.1}$, find a bound for the error in the approximation.

    Answer: $P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

    $\sqrt{1.1} \approx P_2(0.1) = 1.04875$

    $|R_2(0.1)| \leq 0.0000625$

13. Write down the first three terms of the sequence $\{a_n\}$. Then determine the convergence or divergence of the sequence.

    (a) $a_n = 2^{n-1}$

    Answer: The sequence $\{1, 2, 4, ...\}$ diverges.

    (b) $a_n = \dfrac{2n^2}{n^2+1}$

    Answer: The sequence $\left\{1, \frac{8}{5}, \frac{9}{5}, ...\right\}$ converges (to 2).

14. Find the n^{th} term of the sequence (assuming that the 'obvious' pattern continues). Then determine whether the sequence converges or diverges.

    (a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

    Answer: The sequence $\left\{\frac{1}{2n}\right\}$ converges (to 0).

    (b) $1, 4, 7, 10, \ldots$

    Answer: The sequence $\{3n-2\}$ diverges.

15. Given a_n, determine the convergence or divergence of both $\{a_n\}$ and $\sum_{n=0}^{\infty} a_n$. If the sequence/series converges, determine what it converges to.

    (a) $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

    Answer: $\{a_n\}$ converges to 0 and $\sum_{n=0}^{\infty} a_n$ converges to 1.

    (b) $a_n = 2e^{-n}$

    Answer: $\{a_n\}$ converges to 0 and $\sum_{n=0}^{\infty} a_n$ converges to $\dfrac{2}{1-\frac{1}{e}} = \dfrac{2e}{e-1}$.

16. Determine whether the series converges or diverges. If it converges, find its sum.

(a) $\displaystyle\sum_{n=0}^{\infty} 4\left(-\frac{2}{3}\right)^n$

Answer: The series converges to $\dfrac{12}{5}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$

Answer: The series diverges.

(c) $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$

Answer: The series $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$ converges to $\dfrac{3}{2}$.