Practice Exam #2 Solutions

#1 $f(x,y) = \sqrt{x^3y+y^2}$

$f_x = \frac{1}{2}(x^3y+y^2)^{-1/2}(3x^2y) = \dfrac{3x^2y}{2\sqrt{x^3y+y^2}}$ | $f_y = \frac{1}{2}(x^3y+y^2)^{-1/2}(x^3+2y) = \dfrac{x^3+2y}{2\sqrt{x^3y+y^2}}$

#2 $f(x,y) = 25x^{1/3}y^{2/3}$ x: Labor, y: Capital. Compute Marginal Productivities @ $x=27, y=64$

Labor: $f_x = \frac{25}{3}x^{-2/3}y^{2/3} \Rightarrow f_x(27,64) = \frac{25}{3}(27)^{-2/3}(64)^{2/3} = \frac{25}{3}\left(\frac{1}{9}\right)(16) = \frac{400}{27}$

Capital: $f_y = \frac{50}{3}x^{1/3}y^{-1/3} \Rightarrow f_y(27,64) = \frac{50}{3}(27)^{1/3}(64)^{-1/3} = \frac{50}{3}(3)\left(\frac{1}{4}\right) = \frac{25}{2}$

So, Gov't should encourage increased investment in Labor.

#3 $R(x,y) = -0.005x^2 - 0.003y^2 - 0.002xy + 20x + 15y$ Maximize Profit

$C(x,y) = 6x + 3y + 200$

Let $P(x,y)$ = Profit. Then $P(x,y) = R(x,y) - C(x,y) = -0.005x^2 - 0.003y^2 - 0.002xy + 14x + 12y - 200$

$P_x = -0.01x - 0.002y + 14$

$P_y = -0.006y - 0.002x + 12$

$\Rightarrow$ $0.01x + 0.002y = 14$, $0.002x + 0.006y = 12$ $\Rightarrow$ $0.03x + 0.006y = 42$, $0.002x + 0.006y = 12$ $\Rightarrow$ $0.028x = 30$ $\Rightarrow x = 1071.43$ $\Rightarrow$ $y = 1642.86$

$\Rightarrow P = 17{,}157.14$

#4 $P(x,y) = -0.02x^2 - 15y^2 + xy + 39x + 25y - 20{,}000$

$x = 4000$, $y = 150$ $dx = 500$, $dy = -10$

$$\begin{aligned} dP = P_x\,dx + P_y\,dy &= (-0.04x + y + 39)dx + (-30y + x + 25)dy \\ &= (-0.04(4000) + 150 + 39)(500) + (-30(150) + 4000 + 25)(-10) \\ &= 14{,}500 + 4{,}750 \\ &= 19{,}250 \end{aligned}$$

#5 $f(x,y) = 8x^2 - 2y$ Constraint $x^2+y^2=1$

$F(x,y,\lambda) = 8x^2 - 2y + \lambda(x^2+y^2-1)$

$F_x = 16x + 2x\lambda$

$F_y = -2 + 2y\lambda$

$F_\lambda = x^2+y^2-1$

$\Rightarrow$ $2x\lambda = -16x$, $2y\lambda = 2$

Case 1: $x=0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ $f(0,1) = -2$, $f(0,-1) = 2$

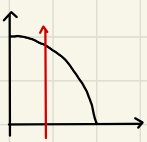
Case 2: $x \neq 0 \Rightarrow \lambda = -8 \Rightarrow y = \frac{1}{\lambda} = -\frac{1}{8} \Rightarrow x^2 + \frac{1}{64} = 1$

$\Rightarrow x^2 = \frac{63}{64}$

$f\left(\pm\frac{\sqrt{63}}{8}, -\frac{1}{8}\right) = 8\left(\frac{63}{64}\right) + \frac{1}{4} = \frac{65}{8}$

So, Minimum: $f(0,1) = -2$

#6 R



$$\begin{aligned} &\int_0^2\int_0^{\sqrt{4-x^2}} y\,dy\,dx \\ &= \int_0^2 \tfrac{1}{2}y^2\Big|_0^{\sqrt{4-x^2}}dx \\ &= \tfrac{1}{2}\int_0^2(4-x^2)dx \\ &= \tfrac{1}{2}\left[4x - \tfrac{x^3}{3}\right]_0^2 \\ &= \tfrac{1}{2}\left[8 - \tfrac{8}{3}\right] \\ &= \tfrac{8}{3} \end{aligned}$$

OR

R

$$\begin{aligned} &\int_0^2\int_0^{\sqrt{4-y^2}} y\,dx\,dy \\ &= \int_0^2 xy\Big|_0^{\sqrt{4-y^2}}dy \\ &= \int_0^2 y\sqrt{4-y^2}\,dy \\ &= -\tfrac{1}{2}\cdot\tfrac{2}{3}(4-y^2)^{3/2}\Big|_0^2 \\ &= \tfrac{1}{3}(4)^{3/2} = \tfrac{8}{3} \end{aligned}$$

#7 SOLVE $y' = \frac{2y+3}{x^2}$

$$\frac{dy}{2y+3} = \frac{dx}{x^2} \Rightarrow \frac{1}{2}\ln(2y+3) = \frac{-1}{x} + D$$

$$\Rightarrow \ln(2y+3) = -\frac{2}{x} + D$$

$$\Rightarrow 2y+3 = Ce^{-2/x}$$

$$\Rightarrow y = -\frac{3}{2} + Ce^{-2/x}$$

CHECK: $y' = \frac{2}{x^2}Ce^{-2/x}$

$$\frac{2y+3}{x^2} = \frac{2\left(-\frac{3}{2} + Ce^{-2/x}\right) + 3}{x^2}$$

$$= \frac{2}{x^2}Ce^{-2/x} \checkmark$$

#8 LET C(t) BE NUMBER OF CLAIMS IN t-th WEEK

$$\frac{dC}{dt} = k(40-C) \quad C(2) = 10, \; C(0) = 0. \; \text{FIND } C(16)$$

$$\frac{dC}{40-C} = k\,dt \Rightarrow -\ln(40-C) = kt + C \qquad C(0)=0 \Rightarrow -\ln 40 = C$$

$$\Rightarrow -\ln(40-C) = kt - \ln 40$$

$$\Rightarrow 40 - C = 40e^{kt}$$

$$\Rightarrow C = 40 - 40e^{kt}$$

$$C(2) = 10 \Rightarrow 10 = 40 - 40e^{2k}$$

$$\Rightarrow e^{2k} = \frac{3}{4}$$

$$\Rightarrow 2k = \ln 3/4$$

$$\Rightarrow k = \frac{1}{2}\ln\frac{3}{4}$$

$$\Rightarrow C(6) = 40 - 40e^{6(\frac{1}{2}\ln 3/4)}$$

$$= 40\left(1 - e^{3\ln 3/4}\right)$$

$$= 40\left(1 - \left(\frac{3}{4}\right)^3\right)$$

$$= 40\left(\frac{37}{64}\right) = \frac{370}{16} = 23.125 \approx 23/\text{Day}$$

#9 $y' = 2x - y + 1 \quad y(0) = 2, \quad 0 \le x \le 2, \quad n = 5 \Rightarrow h = \frac{2-0}{5} = 0.4$

$y_0 = 2$

$y_1 = 2 + (2(0) - 2 + 1)(0.4) = 1.6$

$y_2 = 1.6 + (2(.4) - 1.6 + 1)(0.4) = 1.68$

$y_3 = 1.68 + (2(.8) - 1.68 + 1)(0.4) = 2.048$

$y_4 = 2.048 + (2(1.2) - 2.048 + 1)(0.4) = 2.5888$

$y_5 = 2.5888 + (2(1.6) - 2.5888 + 1)(0.4) = \underline{\underline{3.23328}}$