Unit 3 Exam Review – Lectures 26 - 31

1. Find each value at which $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2$ has a relative maximum or minimum. Find the absolute extrema of $f(x)$ on $[-2, 1]$.

$f'(x) = x^3 - x^2 - 2x$

$0 = x(x^2 - x - 2) = x(x-2)(x+1) \quad x = -1, 0, 2$

relative max at $x = 0$

min at $x = -1, 2$

on $[-2, 1]$ there is a critical point at $x = -1, 0$

$f(-2) = 8/3$

$f(-1) = -\frac{5}{12}$

$f(0) = 0$

$f(1) = -\frac{13}{12}$

absolute max on $[-2,1]$ $\left(-2, \frac{8}{3}\right)$

absolute min $\left(1, -\frac{13}{12}\right)$

2. Find all critical numbers and relative extrema of $g(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$. $= \frac{1}{2}x^{-\frac{1}{2}} - x^{-2}$

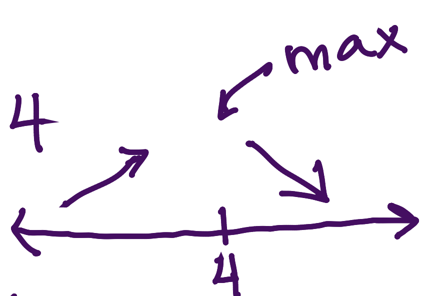
$g'(x) = -\frac{1}{4}x^{-\frac{3}{2}} + 2x^{-3} = 0$

$0 = -4x^3 \cdot \left(-\frac{1}{4}x^{-\frac{3}{2}} + 2x^{-3}\right)$

$0 = x^{3/2} - 8$

$8 = x^{3/2}$

$x = 8^{2/3} = 4$



$g(4) = \frac{1}{2\sqrt{4}} - \frac{1}{16}$

$= \frac{1}{4} - \frac{1}{16}$

$= -\frac{3}{16}$

relative max $\left(4, -\frac{3}{16}\right)$

asymptote at $x=0$

3. Let $f(x) = \dfrac{\sqrt[3]{3x-2}}{x}$. $= \dfrac{(3x-2)^{\frac{1}{3}}}{x}$

(a) Find $f'(x)$ and write as a single fraction.

(b) Find the equation of each horizontal and vertical tangent line of $f(x)$.

(c) Find each x-value at which $f(x)$ has a critical number.

(d) Find the relative extreme values of $f(x)$.

$$f'(x) = \frac{\cancel{\tfrac{1}{3}}(3x-2)^{-2/3}(\cancel{3})x - (3x-2)^{1/3}}{x^2}$$

$$0 = \frac{\dfrac{x}{(3x-2)^{2/3}} - (3x-2)^{1/3}}{x^2} \cdot \frac{(3x-2)^{\frac{2}{3}}}{(3x-2)^{\frac{2}{3}}}$$

$$= \frac{x-(3x-2)}{x(3x-2)^{2/3}} = \frac{-2x+2}{x(3x-2)^{2/3}} = 0 \qquad -2x=-2 \quad x=1$$

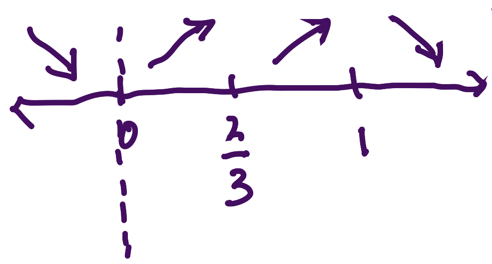
$(3x-2)^{2/3} = 0$ when $x = \frac{2}{3}$

horizontal tangent line at $x=1$ $\quad f(1) = \dfrac{\sqrt[3]{3-2}}{1} = 1$

$y=1$

vertical tangent line $x = \frac{2}{3}$

critical numbers at $x = 1, \frac{2}{3}$



relative max at $x=1$

4. The cost function for a product is
$C(x) = 1.25x^2 + 25x + 8000.$

(a) Suppose the company decides to increase production by 4 units per day when the current daily production level is 50 units. Find the rate of change of cost with respect to time.

$\frac{dx}{dt} = 4$

when $x=50$

$$\frac{dC}{dt} = \left(\frac{5}{4} \cdot 2x + 25\right)\frac{dx}{dt}$$

$$\frac{dC}{dt} = \left(\frac{5}{2}(50) + 25\right)(4)$$

$$= 150 \cdot 4 = 600$$

(b) Find the marginal cost when 50 units are produced. What does it tell you?

$$C'(x) = \frac{5}{2}x + 25$$

$$C'(50) = 150$$

(c) If $C(x) = 1.25x^2 + 25x + 8000$, find each interval on which **average cost** is increasing and decreasing. For what production level x is average cost minimized?

Average cost $= \bar{C}(x) = \frac{C(x)}{x} = \frac{1.25x^2 + 25x + 8000}{x}$

$$\bar{C}(x) = \frac{5}{4}x + 25 + \frac{8000}{x}$$

$$\bar{C}'(x) = \frac{5}{4} - \frac{8000}{x^2} = 0$$

$$\frac{8000}{x^2} = \frac{5}{4}$$

5. The demand function for a certain product is given by
$p(x) = -0.02x + 400$, $0 \leq x \leq 20,000$, where p is the unit price when x items are sold. The cost function for the product is
$C(x) = 100x + 300,000$.

(a) Find the marginal profit of the product when $x = 2000$.

Profit = Revenue − Cost
$= x\,p(x) - C(x)$
$= x(-.02x + 400) - (100x + 300{,}000)$
$P(x) = -.02x^2 + 300x - 300{,}000$
$P'(x) = -.04x + 300$

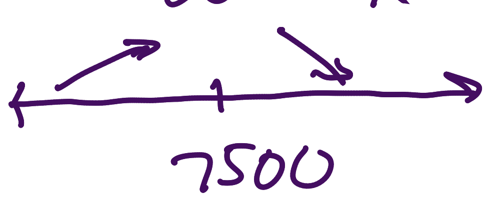
$P'(2000) = -.04(2000) + 300$
$= 220$

(b) Find the actual profit from the sale of the 2001st item. Compare to your answer in (a).

$P(2001) - P(2000) = 219.98$

(c) Find each interval on which the profit function $P(x) = -0.02x^2 + 300x - 300,000$ is increasing and decreasing. Remember that $0 \leq x \leq 20,000$. How many items should be sold to maximize profit? At what price?

$0 = .04x + 300$ $\quad x = 7500$



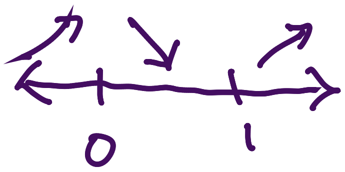
increasing $(0, 7500)$
decreasing $(7500, 20{,}000)$

$p(7500) = -.02(7500) + 400 = 250$

6. Find all relative extrema of $f(x) = 2x^{5/3} - 5x^{2/3}$. Then find the absolute extrema of $f(x)$ on $[-8, 0]$. Compare the two methods.

$f'(x) = \frac{10}{3} x^{\frac{2}{3}} - \frac{10}{3} x^{-\frac{1}{3}} = 0$

$x - 1 = 0 \quad x = 1 \qquad$ critical points at $x = 0, 1$



relative max $f(0) = 0$ $\qquad f(-8) = -84$ absolute min

min $f(1) = -3$ $\qquad f(0) = 0$ absolute max

7. Find the absolute maximum and minimum values of $f(x) = e^{x^3 - 12x}$ on $[0, 3]$.

$f'(x) = (3x^2 - 12)e^{x^3 - 12x} = 0 \qquad 3x^2 - 12 = 0 \quad x^2 = 4 \quad x = \pm 2$

$f(0) = 1$ max

$f(2) = e^{-16}$ min

$f(3) = e^{-9}$

8. Find the maximum and minimum values of $f(x) = x^2 - 8\ln x$ on $[1, e]$.

$f'(x) = 2x - \frac{8}{x} = 0 \qquad 2x^2 = 8$

$x = \pm 2$

$f(1) = 1$ max

$f(2) = 4 - 8\ln(2)$ min

$f(e) = e^2 - 8$

9. The position (in centimeters) of a particle moving in a straight line at time t (in seconds) is given by $s(t) = t^3 - 6t^2 + 9t$ for $0 \le t \le 6$.

(a) Find the velocity function $v(t)$.

(b) At what time(s) is the particle at rest?

(c) For what time interval(s) over the first six seconds is the particle traveling in a positive direction?

(d) Find the average velocity from $t = 0$ to $t = 4$ seconds.

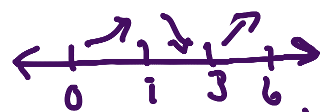
(e) What is the acceleration of the particle after 3/2 second? Include units in your answer.

(f) Find each interval on which the particle is (1) speeding up and (2) slowing down

a)
$v(t) = s'(t) = 3t^2 - 12t + 9$

b) particle is at rest when $v(t) = 0$

c) $0 = 3(t - 4t + 3) = 3(t-3)(t-1)$



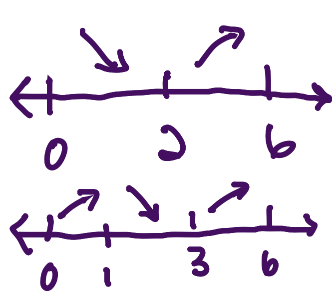
traveling in a positive direction
$(0,1)$ and $(3,6)$

d) average velocity $= \dfrac{s(4) - s(0)}{4 - 0} = \dfrac{4}{4} = 1$

e) $a(t) = v'(t) = s''(t) = 6t - 12$

$a\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right) - 12 = -3$ cm/sec^2

f) $a(t)$



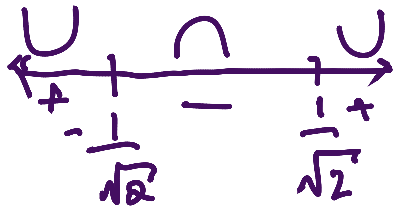
speeding up $(1,2)$, $(3,6)$

inflection point at $x = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

10. Find each interval on which $f(x) = e^{1-x^2}$ is concave up and down, and find each inflection point of the graph of f.

$f'(x) = -2x e^{1-x^2}$

$f''(x) = -2e^{1-x^2} + 4x^2 = 0 = -2e^{1-x^2}(1-2x^2)$ $\quad x = \pm\frac{1}{\sqrt{2}}$



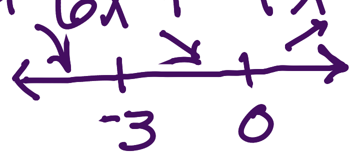
concave up $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

concave down $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

11. Find all intervals on which the graph of
$f(x) = \dfrac{x^4}{4} + 2x^3 + \dfrac{9}{2}x^2 + 8$ is both decreasing and concave up.

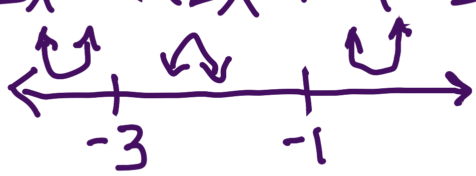
$f'(x) = x^3 + 6x^2 + 9x = x(x^2+6x+9) = x(x+3)^2$

$x = 0, -3$



$f''(x) = 3x^2 + 12x + 9 = 3(x^2+4x+3) = 3(x+3)(x+1)$

$x = -1, -3$



$(-\infty, -3), (-1, 0)$

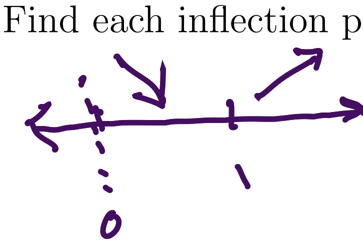
12. Find each interval on which $f(x) = \ln x + \dfrac{1}{x}$ is both increasing and concave down. Find each inflection point.

domain $(0, \infty)$

$x^2 \cdot f'(x) = \frac{1}{x} - \frac{1}{x^2} = 0 \cdot x^2$

$x - 1 = 0$

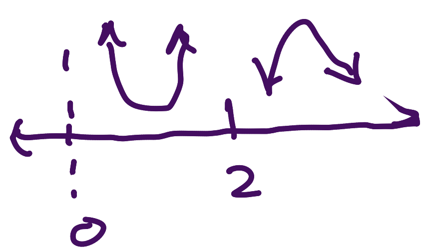
$x^3 \; x = 1$



$f''(x) = -\frac{1}{x^2} + \frac{2}{x^3} = 0 \cdot x^3$

$-x + 2 = 0$

$x = 2$



$(2, \infty)$

inflection point $f(2) = \ln 2 + \frac{1}{2}$ $\quad (2, \ln 2 + \frac{1}{2})$

$f'=0$ at $x=-2,1,5$

13. Suppose that $\boxed{f}(x)$ has horizontal tangent lines at $x=-2$, $x=1$ and $x=5$. If $f''(x)>0$ on intervals $(-\infty,0)$ and $(2,\infty)$ and $f''(x)<0$ on the interval $(0,2)$, find the x- values at which $f(x)$ has relative extrema. Assume that f and f' are continuous on $(-\infty,\infty)$ and use the Second Derivative test.

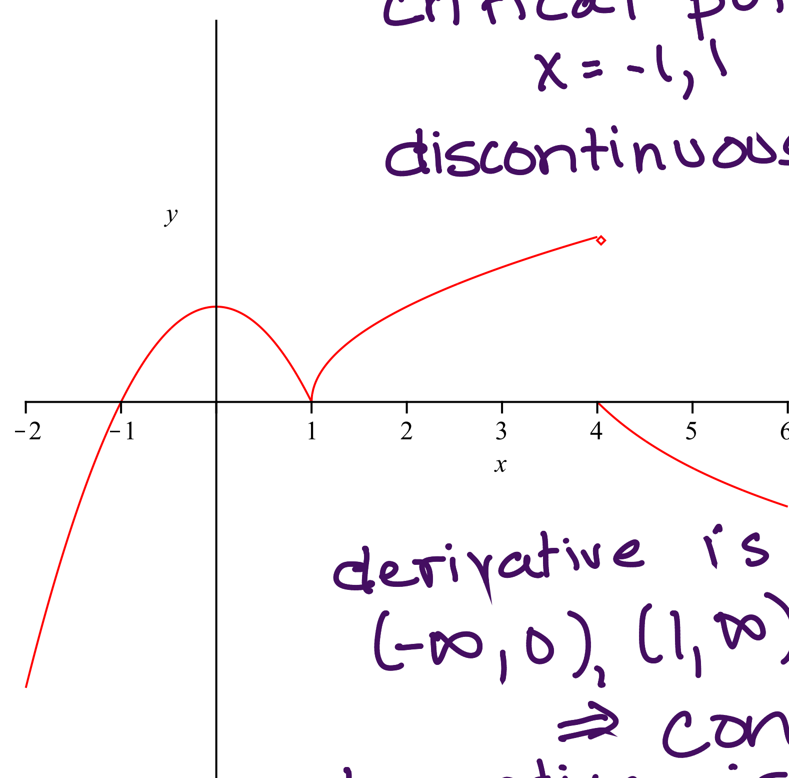
$f''(-2)>0 \Rightarrow$ concave up $\Rightarrow$ relative min

$f''(1)<0 \Rightarrow$ concave down $\Rightarrow$ relative max

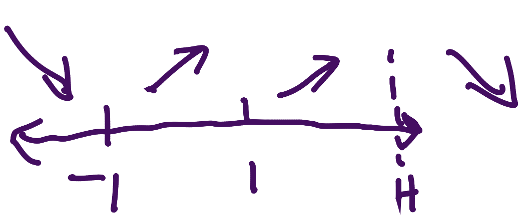
$f''(5)>0 \Rightarrow$ concave up $\Rightarrow$ relative min

14. Given the graph of **the derivative** $f'(x)$, find each interval on which the function $f(x)$ is increasing and decreasing, and find the x-coordinate of each point at which $f(x)$ has a local maximum or minimum value. Find each interval on which $f(x)$ is concave up and down, and the x-coordinate of each inflection point. Assume that the domain of $f(x)$ is $(-\infty,\infty)$.

critical points at
$x=-1,1$
discontinuous at $x=4$



derivative is increasing
$(-\infty,0),(1,\infty)$
$\Rightarrow$ concave up
derivative is decreasing
$(0,1),(4,\infty) \Rightarrow$ concave down

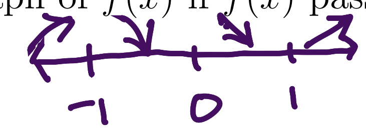


relative min at $x=-1$

15. Suppose that $f'(x) = 15x^4 - 15x^2$. Find each x-value at which the function $f(x)$ has relative extrema. Find the x-coordinate of each inflection point. Sketch a possible graph of $f(x)$ if $f(x)$ passes through the origin.

$0 = 15x^2(x^2-1)$

$x = \pm 1, 0$



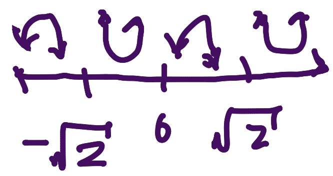
relative max at $x=-1$

min at $x=1$

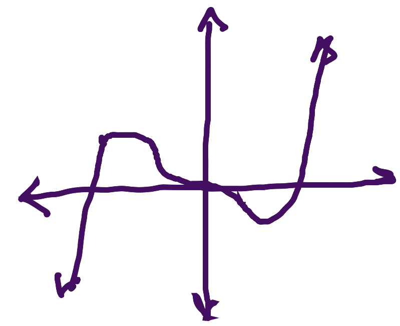
$f''(x) = 60x^3 - 30x$

$= 30x(x^2-2)$

$x = \pm\sqrt{2}, 0$



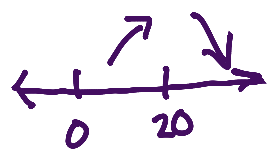
points of inflection at $x = \pm\sqrt{2}, 0$



16. A drug that stimulates reproduction is introduced into a population of viruses. That population can be modeled by the function $P(t) = 30t^2 - t^3 + 200$, $0 \leq t \leq 30$, where $P(t)$ is the population after t minutes.

(a) At what time does the population reach its maximum? What is the maximum population?

$P'(t) = 60t - 3t^2 = 3t(20-t)$



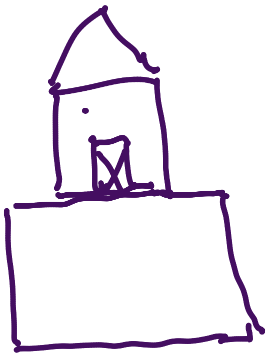
at $t=20$ the population reaches its maximum

(b) At what time is the rate of growth of the population maximized?

$P''(t) = 60 - 6t = 6(10-t)$

at $t=10$

17. A farmer wishes to fence an area next to his barn. He needs a wire fence that costs \$1 per linear foot in front of the barn and and wooden fencing that costs \$2 per foot on the other sides. Find the lengths x (sides perpendicular to the barn) and y (side across from the barn) so that he can enclose the maximum area if his budget for materials is \$4400.



$1x + 2x + 2y + 2y = 4400$ $\qquad$ $3x + 4y = 4400$

$y = 1100 - \frac{3}{4}x$

$A(x) = xy$

$= x\left(1100 - \frac{3}{4}x\right)$

$= 1100x - \frac{3}{4}x^2$

$A'(x) = 1100 - \frac{3}{2}x = 0$

$\frac{3}{2}x = 1100$ $\qquad$ $x = \frac{2200}{3}$ $\qquad$ $y = 1100 - \frac{3}{4} \cdot \frac{2200}{3}$

$= 550$

$(50, 40)$, $(55, 38)$

18. Frye's Electronics has started selling a new video game. In one of its Dallas stores, an average of 50 games sell per month at the regular price of \$40. The manager of the department has observed that when the video game is put on sale, an average of 5 more games will sell for each \$2 price decrease. If each video game costs the store \$24 and there are fixed costs of \$5600, how many should be sold in a given month to maximize profit? What price should they charge? Assume that the demand function $p(x)$ is linear.

$m = \frac{\Delta p}{\Delta x} = \frac{-2}{5}$

$p(x) = mx + b$

$40 = -\frac{2}{5}(50) + b$

$40 = -20 + b \quad b = 60$

$p(x) = -\frac{2}{5}x + 60$

cost = fixed cost + 24x

$C(x) = 3600 + 24x$

Profit = Revenue − cost

$P(x) = xp(x) - C(x)$

$= x\left(-\frac{2}{5}x + 60\right) - (3600 + 24x)$

$= -\frac{2}{5}x^2 + 36x - 3600$

$P'(x) = -\frac{4}{5}x + 36$

$0 = -\frac{4}{5}x + 36$

$\frac{4}{5}x = 36$

$= 45$ will maximize profit

$p(45) = -\frac{2}{5}(45) + 60$

$= 42$

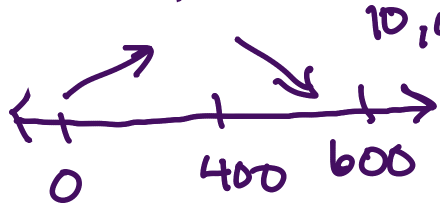
they should charge \$42

19. The revenue $R(x)$ generated from sales of a certain product is related to the amount of money spent on advertising according to the model $R(x) = \frac{1}{10,000}(600x^2 - x^3)$, $0 \le x \le 600$, where x and $R(x)$ are measured in thousands of dollars. Find each interval over which $R(x)$ is increasing. For the interval on which $R(x)$ is increasing, find the point of diminishing returns. Why is it significant?

$R'(x) = \frac{1}{10,000}(1200x - 3x^2) = \frac{1}{10,000}\, x\,(1200 - 3x)$

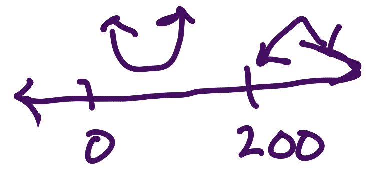
$3x = 1200$

$x = 400$



increasing on $(0,400)$

$R''(x) = \frac{1}{10,000}(1200 - 6x)$



$6x = 1200$

$x = 200$

diminishing returns on $(200,600)$

↑ point of inflection, point of diminishing returns

20. Consider the function $f(x) = x^{1/3}(x+3)$ and its first two derivatives,

$f'(x) = \frac{4x+3}{3x^{2/3}}$ and $f''(x) = \frac{4x-6}{9x^{5/3}}$.

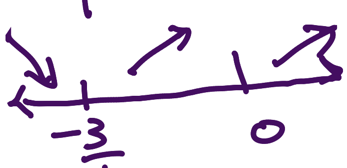
Find all intercepts, asymptotes, relative extrema and inflection points. Sketch the graph of $f(x)$.

x-intercepts $(0,0)$, $(-3,0)$ no asymptotes

y-intercept $(0,0)$

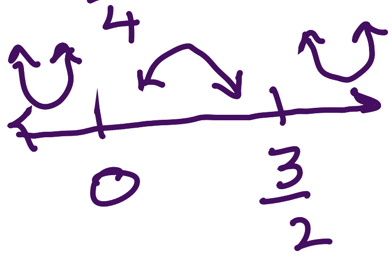
vertical change line

$f'(x) = 0 = \frac{4x+3}{3x^{2/3}}$ critical points at $x = 0, -\frac{3}{4}$

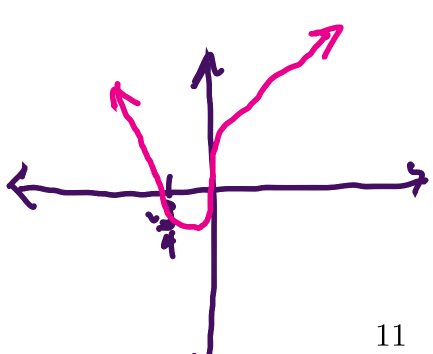


relative min at $x = -\frac{3}{4}$

$f''(x) = 0 = \frac{4x-6}{9x^{5/3}}$



$x = \frac{3}{2}$

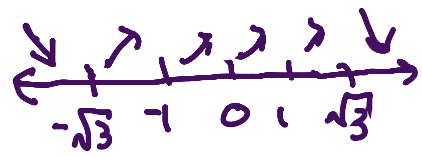


x,y - intercept (0,0)
asymptotes $x = \pm 1$

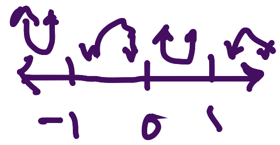
21. Sketch the graph of $f(x) = \dfrac{x^3}{1-x^2}$ if

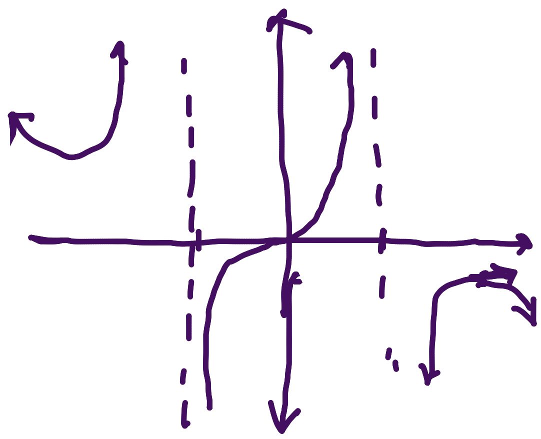
$$f'(x) = \frac{x^2(3-x^2)}{(1-x^2)^2} \text{ and } f''(x) = \frac{2x(x^2+3)}{(1-x^2)^3}.$$

$f'(x) = \dfrac{x^2(3-x^2)}{(1-x^2)^2} = 0$ critical points $x = 0, \pm\sqrt{3}$

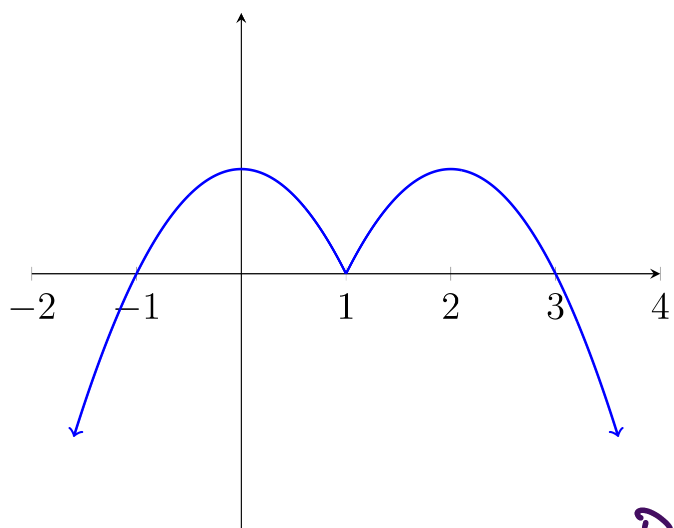


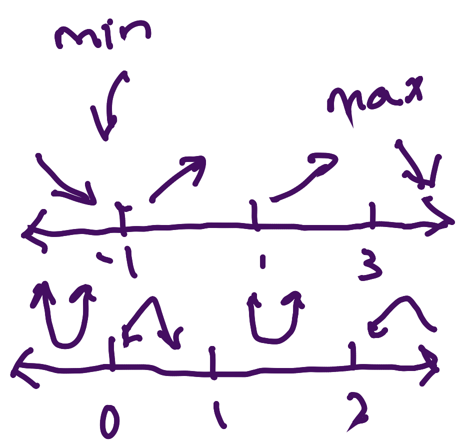
$f''(x) = \dfrac{2x(x^2+3)}{(1-x^2)^3} = 0$ $x = 0 \leftarrow$ point of inflection



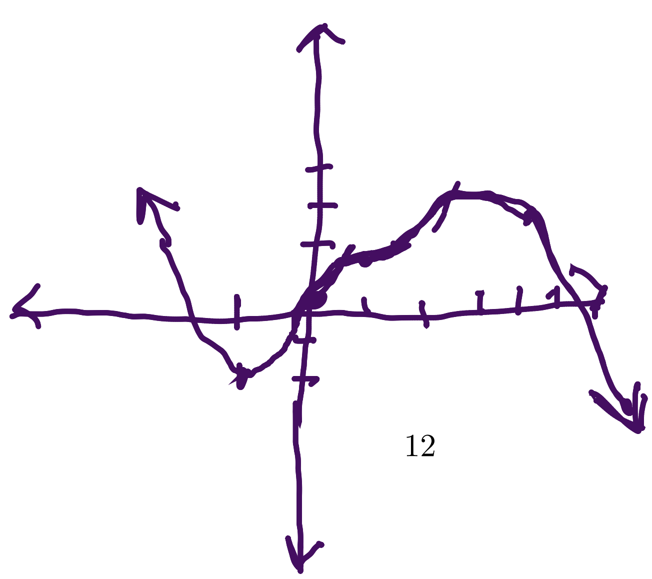


22. Given the graph of the derivative $f'(x)$, find a possible graph of the function $f(x)$. Assume that $f(-1) = -2$, $f(0) = 0$, $f(1) = 1$, $f(2) = 3$ and $f(3) = 5$, and that $f(x)$ is a continuous function. Be sure to find all extrema and inflection points.





points of inflection at $x = 0, 1, 2$



MAC 2233: Review: Lectures 30 to 34

Part I: Multiple Choice

1. The slope of a curve $y = f(x)$ at any point is given by $f'(x) = \dfrac{6x-1}{\sqrt{x}}$. If the curve passes through the point $(1,1)$, find $f(4)$.

a. 29 b. $13 - \ln 4$ c. 27 d. $\dfrac{11}{2}$ e. 28

$f(x) = \int f'(x)\,dx = \int \dfrac{6x-1}{x^{\frac{1}{2}}}\,dx$

$= \int 6x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,dx$

$f(x) = 6 \cdot \frac{2}{3} x^{3/2} - 2x^{\frac{1}{2}} + C$

$f(1) = 4(1) - 2(1) + C = 1 \Rightarrow C = -1$

$f(x) = 4x^{3/2} - 2x^{1/2} - 1$

$f(4) = 4 \cdot 4^{3/2} - 2 \cdot 4^{1/2} - 1 = 32 - 4 - 1 = 27$

2. If $f'(x) = \dfrac{(\ln x)^2}{x}$ and $f(e) = -1$, find $f(e^4)$.

a. $\dfrac{1}{3}$ b. $\ln 4$ c. $\dfrac{16}{e^4}$ d. $\dfrac{60}{3}$ e. 4

$f(x) = \int \dfrac{(\ln x)^2}{x}\,dx$ $u = \ln x$, $du = \frac{1}{x}dx$

$\int u^2\,du = \dfrac{u^3}{3} + C$

$f(x) = \dfrac{(\ln x)^3}{3} + C$

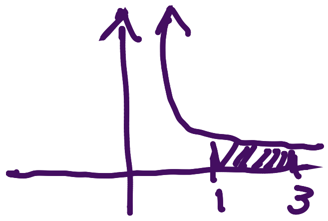
$f(e) = \dfrac{(\ln e)^3}{3} + C = -1$

$\frac{1}{3} + C = -1$ $C = -\frac{4}{3}$

$f(x) = \dfrac{(\ln x)^3}{3} - \dfrac{4}{3}$

$f(e^4) = \dfrac{(\ln e^4)^3}{3} - \dfrac{4}{3}$

$= \dfrac{4^3}{3} - \dfrac{4}{3} = \dfrac{60}{3}$



3. Let $f(x) = \frac{2}{x}$. What is the exact area of the region bounded by $f(x)$ and the x-axis from $x = 1$ to $x = 3$?

a. $\ln 9$ b. $\frac{\ln 3}{2} - \frac{1}{2}$ c. $\ln 3$ d. $\ln 9 - 2$ e. 9

$$\int_1^3 \frac{2}{x}\,dx = 2\ln|x|\Big|_1^3$$
$$= 2\ln|3| - \overset{0}{2\ln|1|}$$
$$= 2\ln(3)$$
$$= \ln 9$$

4. The marginal revenue and cost functions for a new product are $R'(x) = 72 - 0.2x$ and $C'(x) = 0.4x$ respectively where x is the number of items sold. Find the profit function $P(x)$ if the developers of the product will lose their initial investment of \$1200 if there are no sales. **Hint:** Profit = Revenue − Cost. What is $P(0)$? $P(0) = -1200$

a. $P(x) = 72 - 0.6x - 1200$

b. $P(x) = 72x + 0.1x^2 - 1200$

c. $P(x) = 72x - 0.3x^2 - 1200$

d. $P(x) = 72x + 0.1x^2 + 1200$

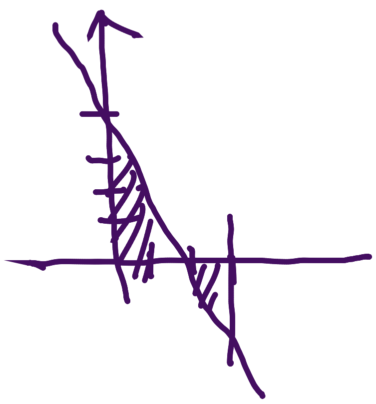
$$P(x) = \int R'(x) - C'(x)\,dx$$
$$= \int 72 - .2x - .4x \quad dx$$
$$= \int 72 - .6x$$
$$= 72x - .3x^2 + C$$
$$P(0) = 0 + C = -1200 \Rightarrow C = -1200$$

$$P(x) = 72x - .3x^2 - 1200$$

$0 = 4 - 2x$
$2x = 4$
$x = 2$

5. Find the area (in square units) of the region(s) bounded by $f(x) = 4 - 2x$ and the x-axis from $x = 0$ to $x = 3$. Be sure to sketch the area.

a. 3 b. 4 c. 5 d. $\frac{7}{2}$ e. 6



$\int_0^2 4-2x\,dx - \int_2^3 4-2x\,dx$

$= 4x - x^2 \Big|_0^2 - (4x - x^2)\Big|_2^3$

$= 4\cdot 2 - 2^2 - (4\cdot 3 - 3^2) + (4\cdot 2 - 2^2)$

$= 8 - 4 - (12 - 9) + 8 - 4$

$= 4 - 3 + 4$

$= 5$

6. Evaluate $\int_0^3 (4 - 2x)\,dx$. Compare to the previous problem.

a. 3 b. 4 c. 5 d. $\frac{7}{2}$ e. 6

$\int_0^3 4-2x\,dx = 4x - x^2 \Big|_0^3$

$= 12 - 9$

$= 3$

7. Find the maximum and minimum values of $f(x) = xe^{-x}$ on $[0, 3]$.

a. $\frac{3}{e^3}$ and $-e$ b. $\frac{1}{e}$ and $\frac{3}{e^3}$ c. $\frac{1}{e}$ and 0

d. $\frac{3}{e^3}$ and 0 e. 0 and $-e$

$f'(x) = e^{-x} - xe^{-x}$

$0 = e^{-x}(1-x) \quad x = 1$

$f(0) = 0$ min

$f(1) = \frac{1}{e}$ max

$f(3) = \frac{3}{e^3}$

8. Evaluate $\int \left(x - \frac{1}{x}\right)^2 dx$. $= \int \left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right) dx$

$= \int x^2 - 2 + \frac{1}{x^2} \, dx$

$= \frac{x^3}{3} - 2x - \frac{1}{x} + C$

a. $\frac{x^3}{3} - \frac{1}{x} + C$

b. $\frac{x^3}{3} + \ln(2x) + C$

c. $\frac{1}{3}\left(x - \frac{1}{x}\right)^3 + C$

d. $\frac{x^3}{3} - 2x - \frac{1}{x} + C$ e. $\frac{x^3}{3} - 2x + \ln(x^2) + C$

Indicate whether each statements is true or false.

9. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval, then $F(4)$ must equal $G(4)$.

   a. True b. False

$$\int e^{6x}\,dx = \frac{1}{6}e^{6x} + C$$
$$= \frac{1}{6}f(x) + C$$

10. If $f(x) = e^{6x}$, then $\int f(x)\,dx = 6f(x) + C$.

    a. True b. False

11. $\int 4^{2x+1}\,dx = \frac{(\ln 4)4^{2x+1}}{2}$.

    a. True b. False

$$\int 4^{2x+1}\,dx = \frac{4^{2x+1}}{\ln(4)(2)} + C$$

12. If $f(3) = -4$, $f'(x)$ is continuous, and $\int_3^6 f'(x)\,dx = 8$, then $f(6) = 4$.

    a. True b. False

$$\int_3^6 f'(x) = f(6) - f(3)$$
$$= 4 - (-4)$$
$$= 8$$

Part II: Work each problem

1. (a) If $f(x) = x^2 + 2x - 3$ find the actual change in f when x changes from 0 to 0.15.

   (b) Find the approximate change in $f(x)$ using differentials.

a) $f(.15) - f(0) = .3225$ $\quad \Delta x = .15 - 0 = .15$

b) $\Delta y \approx f'(x)\Delta x$ $\quad f'(x) = 2x + 2$

$= 2(.15)$ $\quad f'(0) = 2$

$= .3$

2. The demand function for a product is $p(x) = 45 - \dfrac{\sqrt{x}}{2}$ where p is the price at which x items will sell.

   (a) Use differentials to approximate the change in revenue when the number of units sold decreases from 1600 to 1590. What is the exact change in revenue?

$R(x) = x\,p(x)$

$= x\left(45 - \dfrac{\sqrt{x}}{2}\right)$

$= 45x - \dfrac{x^{3/2}}{2}$

$R'(x) = 45 - \frac{3}{4}x^{\frac{1}{2}}$

$R'(1600) = 45 - \frac{3}{4}(40)$

$= 15$

$\Delta x = 1590 - 1600$

$\Delta R \approx R'(1600)\Delta x$

$= 15(-10)$

$= -150$

$\Delta R = R(1590) - R(1600)$

$= -150.47$

(b) Now suppose that the number of units sold is increasing by 20 per week. At what rate is revenue changing with respect to time at a production level of 900 units?

$\frac{dx}{dt} = 20$

$$\frac{dR}{dt} = \left(45 - \frac{3}{4}x^{\frac{1}{2}}\right)\frac{dx}{dt}$$

$$= \left(45 - \frac{3}{4}(900)^{\frac{1}{2}}\right)(20)$$

$$= 450$$

Revenue is increasing by \$450/week

(c) What price should be charged for the product in order to maximize revenue?

$$R'(x) = 45 - \frac{3}{4}x^{\frac{1}{2}} = 0$$

$$\frac{3}{4}x^{\frac{1}{2}} = 45$$

$$x^{\frac{1}{2}} = 60$$

$$x = 3600$$

$$p(3600) = 45 - \frac{\sqrt{3600}}{2} = 15$$

sell 3600 at a price of \$15

3. Find $f(x)$ so that $\int f(x)\,dx = (1 + \ln x)^2 + C$

$$f(x) = \left[(1 + \ln x)^2 + C\right]'$$

$$= 2(1 + \ln x)\frac{1}{x}$$

4. Evaluate each integral:

a) $\int_3^8 \frac{3x}{x^2-4}\,dx$ b) $\int \frac{(\sqrt{x}+1)^2}{x}\,dx$ c) $\int_1^e \frac{(1+2\ln x)^2}{x}\,dx$

d) $\int \frac{e^{\frac{1}{2x-1}}}{(2x-1)^2}\,dx$ e) $\int_0^4 \sqrt{3x+4}\,dx$ f) $\int_0^2 xe^{3x^2}\,dx$

g) $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$

a) $\int_3^8 \frac{3x}{x^2-4}\,dx$

$u = x^2-4$

$du = 2x\,dx$

$\frac{3}{2}du = 3x\,dx$

$\frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|u|\Big|_{x=3}^{x=8}$

$= \frac{3}{2}\ln|x^2-4|\Big|_3^8$

$= \frac{3}{2}\ln(60) - \frac{3}{2}\ln|5|$

b) $\int \frac{(\sqrt{x}+1)^2}{x}\,dx = \int \frac{x+2\sqrt{x}+1}{x}\,dx$

$= \int 1 + 2x^{-\frac{1}{2}} + \frac{1}{x}\,dx$

$= x + 2\cdot\frac{2}{3}x^{\frac{3}{2}} + \ln|x| + C$

$= x + \frac{4}{3}x^{3/2} + \ln|x| + C$

c) $\int_1^e \frac{(1+2\ln x)^2}{x}\,dx$

$u = 1+2\ln x$

$du = \frac{1}{x}\,dx$

$\int u^2\,du = \frac{u^3}{3}\Big|_{x=1}^{x=e}$

$= \frac{(1+2\ln x)^3}{3}\Big|_1^e$

$= \frac{(1+2\ln(e))^3}{3} - \frac{(1+2\ln(1))^3}{3}$

$= \frac{3^3}{3} - \frac{1}{3}$

d) $\displaystyle\int \frac{e^{\frac{1}{2x-1}}}{(2x-1)^2}\,dx$

$u = \dfrac{1}{2x-1} = (2x-1)^{-1}$

$du = -(2x-1)^{-2}(2)\,dx$

$-\dfrac{1}{2}\,du = \dfrac{1}{(2x-1)^2}\,dx$

$= -\dfrac{1}{2}\displaystyle\int e^u\,du = -\dfrac{1}{2}e^u + C$

$= -\dfrac{1}{2}e^{\frac{1}{2x+1}} + C$

e) $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$

$u = 3x+4$

$du = 3\,dx$

$\dfrac{1}{3}du = dx$

$\dfrac{1}{3}\displaystyle\int_{x=0}^{x=4} u^{\frac{1}{2}}\,du = \dfrac{2}{9}u^{\frac{3}{2}}\Big|_{x=0}^{x=4}$

$= \dfrac{2}{9}(3x+4)^{\frac{3}{2}}\Big|_0^4$

$= \dfrac{2}{9}(16)^{3/2} - \dfrac{2}{3}(4)^{\frac{3}{2}}$

$= \dfrac{2}{9}(64-8)$

$= \dfrac{2}{9}(56)$

$= \dfrac{112}{9}$

f) $\displaystyle\int_0^2 xe^{3x^2}\,dx$

$u = 3x^2$

$du = 6x\,dx$

$\dfrac{1}{6}du = x\,dx$

$= \dfrac{1}{6}\displaystyle\int_{x=0}^{x=2} e^u\,du$

$= \dfrac{1}{6}e^u\Big|_{x=0}^{x=2}$

$= \dfrac{1}{6}e^{3x^2}\Big|_0^2$

$= \dfrac{1}{6}(e^{12}-1)$

g) $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$ $\quad u = x-1 \Rightarrow u+1 = x$

$du = dx$

$$\int_{x=2}^{x=5} \frac{u+1}{\sqrt{u}}\,du = \int_{x=2}^{x=5} u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \Big|_{x=2}^{5}$$

$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} \Big|_2^5$$

$$= \frac{2}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}} - \left(\frac{2}{3}(1)^{\frac{3}{2}} + 2(1)\right)$$

$$= \frac{2}{3}\cdot 8 + 2\cdot 2 - \frac{2}{3} - 2$$

$$= \frac{14}{3} + 2$$

$$= \frac{20}{3}$$

5. It is estimated that humans are consuming zinc at the rate $R'(t) = 15e^{0.06t}$ million metric tons per year, with $t = 0$ in 2012. If 10 million metric tons were consumed in 2012, find a formula $R(t)$, the amount of zinc consumed in year t.

$$R(t) = \int 15e^{.06t}\,dt$$

$$= 250e^{.06t} + C \qquad R(t) = 250e^{.06t} - 240$$

$$R(0) = 250 + C = 10$$

$$C = -240$$

6. The marginal revenue for a product is $100 + 0.4x - 0.3x^2$. If the revenue from the sale of 20 items is \$1280, find the revenue and demand functions for the product.

$$R(20) = 1280$$

$$R'(x) = 100 + .4x - .3x^2$$

$$R(x) = \int 100 + .4x - .3x^2\,dx$$

$$= 100x + .2x^2 - .1x^3 + C$$

$$R(20) = 100(20) + .2(20)^2 - .1(20)^3 + C = 1280$$

$$1280 + C = 1280$$

$$C = 0$$

$$R(x) = 100x + .2x^2 - .1x^3$$

$v(3) = \frac{3}{2}$

$s(0) = 0$

7. The acceleration of an object is given by $a(t) = \dfrac{2}{(t+1)^2}$ cm/sec^2. If the velocity after 3 seconds is $\dfrac{3}{2}$ cm/sec, find the displacement of the object from its starting point after the first two seconds.

$v(t) = \int \frac{2}{(t+1)^2}\,dt$ $\quad u = t+1$, $du = dt$

$\int 2u^{-2}\,du = -2u^{-1} + C$

$v(t) = -\frac{2}{t+1} + C$

$v(3) = -\frac{2}{4} + C = \frac{3}{2}$ $\quad C = 2$

$s(t) = \int -\frac{2}{t+1} + 2\,dt$

$= -2\ln|t+1| + 2t + C$

$s(0) = -2 \cdot 0 + 0 + C = 0$

$s(t) = -2\ln|t+1| + 2t$

$s(2) = -2\ln|3| + 4$

8. The price of a particular model of a Toyota is increasing at the rate of $\dfrac{3t}{\sqrt{3t^2+4}}$ thousand dollars t years after its introduction. If the retail price of the car when it was first introduced was \$24,000, find the retail price of that same model two years later.

$p'(t) = \frac{3t}{\sqrt{3t^2+4}}$ $\quad p(0) = 24{,}000$ $\quad p(2) = ?$

$p(t) = \int \frac{3t}{\sqrt{3t^2+4}}\,dt$

$u = 3t^2 + 4$

$du = 6t\,dt$

$\frac{1}{2}du = 3t\,dt$

$\frac{1}{2}\int u^{-\frac{1}{2}}\,du = u^{\frac{1}{2}} + C$

$p(t) = \sqrt{3t^2+4} + C$

$p(0) = 2 + C = 24{,}000$ $\quad C = 23{,}998$

9. A car moving along a straight track has acceleration function $a(t) = e^{2t}$ m/sec^2. The initial velocity is 3 m/sec and the initial distance from an observer is 2 meters. Find its position function $s(t)$ which gives the position of the car from the observer after t seconds .

$$v(t) = \int e^{2t}\,dt = \frac{1}{2}e^{2t} + c$$

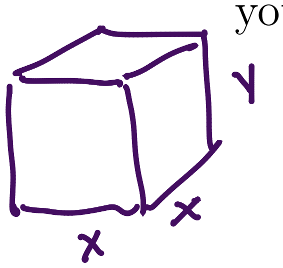
$$v(0) = \frac{1}{2} + c = 3 \qquad c = \frac{5}{2} \qquad v(t) = \frac{1}{2}e^{2t} + \frac{5}{2}$$

$$s(t) = \int \frac{1}{2}e^{2t} + \frac{5}{2}\,dt = \frac{1}{4}e^{2t} + \frac{5}{2}t + c$$

$$s(0) = \frac{1}{4} + c = 2 \qquad s(t) = \frac{1}{4}e^{2t} + \frac{5}{2}t + \frac{7}{4}$$

$$c = \frac{7}{4}$$

10. A rectangular shipping crate is to be constructed with a square base. The material for the two square ends costs \$3 per square foot and the material for the sides costs \$2 per square foot. What dimensions will minimize the cost of constructing the crate if it must have a volume of 12 cubic feet? What is the minimum cost? Let x be the length of the side of a square end, and y be the height of the crate. Be sure to check your answer.



$$V = x \cdot x \cdot y = 12$$

$$x^2 y = 12$$

$$y = \frac{12}{x^2}$$

$$M = 3 \cdot 2x^2 + 2 \cdot 4xy$$

$$= 6x^2 + 8x\left(\frac{12}{x^2}\right)$$

$$= 6x^2 + \frac{96}{x}$$

$$y = \frac{12}{2^2} = 3$$

$$M' = 12x - \frac{96}{x^2}$$

$$12x^3 = 96$$

$$x = 2$$

$$\boxed{\begin{array}{c} x = 2 \\ y = 3 \end{array}}$$

$$M = 6 \cdot 4 + \frac{96}{2} = \$72$$

11. A sporting goods store has started selling a new fitness tracker. In one of its local districts, an average of 50 trackers sell per month at the regular price of \$40. The financial manager has observed that when the tracker is put on sale, an average of 5 more will sell for each \$2 price decrease. If each unit costs the store \$24 and there are fixed costs of \$5600, how many should be sold in a given month to maximize profit? What price should they charge? Assume that the demand function $p(x)$ is linear.

$(50, 40)$

$m = \frac{\Delta p}{\Delta x} = -\frac{2}{5}$

Cost $= 24x + 5600$

$p(x) = -\frac{2}{5}x + 60$

$y = mx + b$

$40 = -\frac{2}{5}(50) + b$

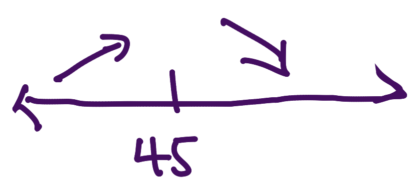
$b = 60$

Profit = Revenue − cost

$P(x) = x\left(-\frac{2}{5}x + 60\right) - (24x + 5600)$

$= -\frac{2}{5}x^2 + 60x - 24x - 5600$

$= -\frac{2}{5}x^2 + 36x - 5600$



$P'(x) = -\frac{4}{5}x + 36$

$\frac{4}{5}x = 36$

$x = 45$

$p(45) = -\frac{2}{5}(45) + 60$

$= \$42$

12. Consider the area given by $\int_0^4 \frac{x}{x^2+1}\,dx$.

(a) Approximate the area using a Riemann sum with $n = 4$ and using the left endpoint of each subinterval to find the height of each rectangle.

(b) Approximate the same area with $n = 4$ using midpoints to approximate the integral.

(c) Find the exact value of the integral.

$x_1 = 0 \quad x_4 = 3$
$x_2 = 1$
$x_3 = 2$

$\Delta x = \frac{4-0}{4} = 1$

a) $\sum_{n=1}^{4} \frac{x_i}{x_i^2+1}\Delta x = \left(\frac{0}{0^2+1} + \frac{1}{1+1} + \frac{2}{4+1} + \frac{3}{9+1}\right)\Delta x$

$= 0 + \frac{1}{2} + \frac{2}{5} + \frac{3}{10}$

$= 1.2$

b) $\sum_{n=1}^{4} \frac{x_i}{x_i^2+1}\Delta x = \frac{\frac{1}{2}}{\frac{1}{4}+1} + \frac{\frac{3}{2}}{\frac{9}{4}+1} + \frac{\frac{5}{2}}{\frac{25}{4}+1} + \frac{\frac{7}{2}}{\frac{49}{4}+1}$

$= \frac{2}{5} + \frac{6}{13} + \frac{10}{29} + \frac{14}{53}$

≈ 1.4705

$x_1 = \frac{1}{2}$
$x_2 = \frac{3}{2}$
$x_3 = \frac{5}{2}$
$x_4 = \frac{7}{2}$

c) $\int_0^4 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{x=0}^{x=4} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_{x=0}^{x=4}$

$= \frac{1}{2}\ln|x^2+1|\Big|_0^4$

$= \frac{1}{2}\ln|17| - \frac{1}{2}\ln|1|$

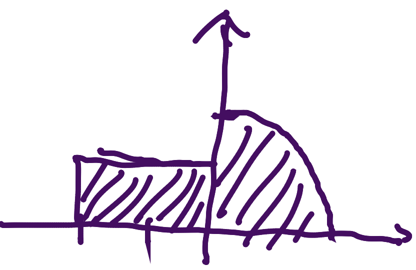
$= \frac{1}{2}\ln|17|$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

13. Find the following definite integral by finding the area of an appropriate geometric figure. Be sure to sketch the regions involved.

$$\int_{-2}^{3} f(x)\,dx \text{ with } f(x) = \begin{cases} 3 & x \le 0 \\ \sqrt{9-x^2} & 0 < x \le 3 \end{cases}$$

$$\int_{-2}^{0} 3\,dx + \int_{0}^{3} \sqrt{9-x^2}$$



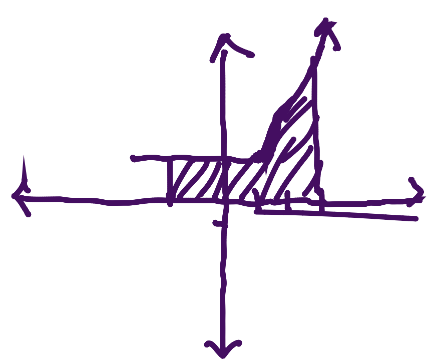
$$= 3 \cdot 2 + \frac{1}{4}\pi(3)^2$$

$$= 6 + \frac{9\pi}{4}$$

14. Find the area of the region bounded by $f(x) = |x-1| + x$ and the x-axis on $[-1, 3]$. Be sure to rewrite the function without absolute value bars, then sketch.

$$f(x) = \begin{cases} x - 1 + x & x - 1 > 0 \\ -x + 1 + x & x - 1 < 0 \end{cases}$$

$$= \begin{cases} 2x - 1 & x > 1 \\ 1 & x < 1 \end{cases}$$



$$\int_{-1}^{1} 1\,dx + \int_{1}^{3} 2x - 1\,dx = x\Big|_{-1}^{1} + x^2 - x\Big|_{1}^{3}$$

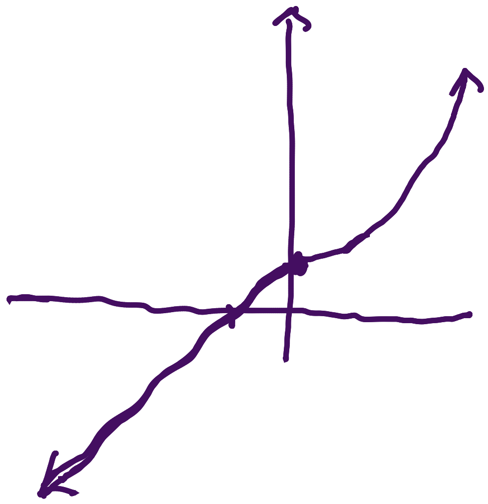
$$= 1 - (-1) + (9 - 3) - (1^2 - 1)$$

$$= 2 + 6$$

$$= 8$$

15. (a) Sketch the graph of the region bounded by the function

$$f(x) = \begin{cases} x+1 & x \leq 0 \\ e^{x/2} & x > 0 \end{cases}$$ and the x-axis from $x = -1$ to $x = 2$.



(b) Find the area of the region described in part (a) by evaluating two separate integrals.

$$A = \int_{-1}^{0} x+1 \, dx + \int_{0}^{2} e^{\frac{x}{2}}$$

$$= \frac{x^2}{2} + x \Big|_{-1}^{0} + 2e^{\frac{x}{2}} \Big|_{0}^{2}$$

$$= 0 - \left(\frac{1}{2} - 1\right) + 2e - 2$$

$$= \frac{1}{2} + 2e - 2$$

$$= 2e - \frac{3}{2}$$

16. Find the values of a and b so that $f(x)$ will be continuous for all x if $f(x) = \begin{cases} x - a & x < 3 \\ 2 & x = 3 \\ \dfrac{x^2}{3} + b & x > 3 \end{cases}$. Then find the area of the region bounded by $f(x)$ and the x-axis on the interval $[1, 4]$.

$$\lim_{x \to 3^-} f(x) = f(3) = 2$$

$$\lim_{x \to 3^-} x - a = 2$$

$$3 - a = 2$$

$$\boxed{a = 1}$$

$$f(x) = \begin{cases} x - 1 & x < 3 \\ \dfrac{x^2}{3} - 1 & x \geq 3 \end{cases}$$

$$\lim_{x \to 3^+} \frac{x^2}{3} + b = f(3) = 2$$

$$\frac{3^2}{3} + b = 2$$

$$\boxed{b = -1}$$

$$A = \int_1^3 x - 1 \, dx + \int_3^4 \frac{x^2}{3} - 1 \, dx$$

$$= \frac{x^2}{2} - x \Big|_1^3 + \frac{x^3}{9} - x \Big|_3^4$$

$$= \frac{9}{2} - 3 - \left(\frac{1}{2} - 1\right) + \left(\frac{64}{9} - 4\right) - \left(\frac{27}{9} - 3\right)$$

$$= \frac{46}{9}$$