The Final exam covers Lectures 1 – 34

MAC 2233: Exam 1 Review
Unit 1 Exam Review covers Lectures 1 – 14

$x \neq -1$

1. Solve for x: $2(x+1)^{-1/3}x^{4/3} - (x+1)^{2/3}x^{-2/3} = 0$

$$x^{\frac{2}{3}}(x+1)^{1/3}\left(\frac{2x^{\frac{4}{3}}}{(x+1)^{1/3}} - \frac{(x+1)^{2/3}}{x^{2/3}}\right) = 0 \cdot x^{2/3}\cdot(x+1)^{1/3}$$

$$2x^{4/3}x^{2/3} - (x+1)^{2/3}(x+1)^{1/3} = 0$$

$$2x^2 - (x+1) = 0$$

$$2x^2 - x - 1 = (2x+1)(x-1) = 0$$

$x = 1, -\frac{1}{2}$

2. Perform the operation and simplify the expression: $\dfrac{\dfrac{3x}{\sqrt{x^2+4}} - \sqrt{x^2+4}}{2\sqrt{x^2+4}}$

$$\frac{\dfrac{3x}{\sqrt{x^2+4}} - \sqrt{x^2+4}}{2\sqrt{x^2+4}} \cdot \frac{\sqrt{x^2+4}}{\sqrt{x^2+4}} = \frac{3x-(x^2+4)}{2(x^2+4)} = \frac{3x-x^2-4}{2(x^2+4)}$$

3. Solve the inequality: $\dfrac{x+4}{x-1} \leq 2$

$$\frac{x+4}{x-1} - 2 \leq 0$$

$$\frac{x+4}{x-1} - \frac{2(x-1)}{x-1} \leq 0$$

$$\frac{x+4-2x+2}{x-1} \leq 0$$

$$\frac{-x+6}{x-1} \leq 0$$

critical numbers
$x = 6, 1$

$-$ 1 $+$ 6 $-$

$(-\infty, -1) \cup [6, \infty)$

4. Find and simplify $\dfrac{f(x+h)-f(x)}{h}$ for

a) $f(x)=2x^2-x-3$ and b) $f(x)=\dfrac{x}{x+4}$.

a) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{2(x+h)^2-(x+h)-3-(2x^2-x-3)}{h}$

$= \dfrac{2(x^2+2xh+h^2)-x-h-3-2x^2+x+3}{h}$

$= \dfrac{\cancel{2x^2}+4xh+2h^2-\cancel{x}-h\cancel{-3}\cancel{-2x^2}\cancel{+x}\cancel{+3}}{h}$

$= \dfrac{4xh+2h^2-h}{h}$

$= \dfrac{\cancel{h}(4x+2h-1)}{\cancel{h}}$

$= 4x+2h-1$

b) $f(x) = \frac{x}{x+4}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+4} - \frac{x}{x+4}}{h} \quad \cdot \frac{(x+4)(x+h+4)}{(x+4)(x+h+4)}$$

$$= \frac{(x+h)(x+4) - x(x+h+4)}{h(x+4)(x+h+4)}$$

$$= \frac{\cancel{x^2} + \cancel{4x} + \cancel{hx} + 4h - \cancel{x^2} - \cancel{xh} - \cancel{4x}}{h(x+4)(x+h+4)}$$

$$= \frac{4\cancel{h}}{\cancel{h}(x+4)(x+h+4)}$$

$$= \frac{4}{(x+4)(x+h+4)}$$

domain of f $(-\infty,2)\cup(2,\infty)$

g $(-\infty,0)\cup(0,\infty)$

5. Let $f(x) = \dfrac{x}{x-2}$ and $g(x) = \dfrac{2}{x} + 1$. Find the functions $(f \circ g)(x)$ and $(g \circ f)(x)$. Include domains.

$x \neq 0, 2$

domain $(-\infty,0)\cup(0,2)\cup(2,\infty)$

$$(f\circ g)(x): (f(g(x)) = \frac{\frac{2}{x}+1}{\frac{2}{x}+1-2} \cdot \frac{x}{x} = \frac{2+x}{2-x}$$

$$(g\circ f)(x) = g(f(x)) = \frac{2}{\frac{x}{x-2}} + 1 = \frac{2(x-2)}{x} + 1 = \frac{2x+4+x}{x} = \frac{3x+4}{x}$$

$x \neq 2, 0$

domain $(-\infty,0)\cup(0,2)\cup(2,\infty)$

6. Let $f(x) = \sqrt{x-1}$ and $g(x) = \dfrac{x}{\sqrt{x-1}}$. Find $\dfrac{f}{g}(x)$ and its domain.

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-1}}{\frac{x}{\sqrt{x-1}}} \cdot \frac{\sqrt{x-1}}{x} = \frac{x-1}{x} \quad x \neq 0$$

$x-1>0$

$x>1$

$(1,\infty)$

$x-1>0$

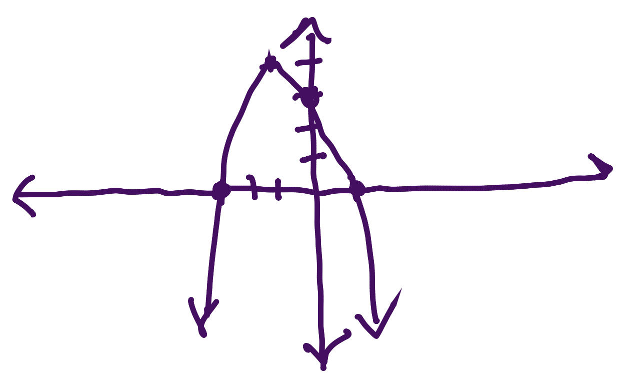
$x^2+2x-3 \quad (x+3)(x-1)$

7. Sketch the graph of $f(x) = 3 - 2x - x^2$ by using a formula to find the vertex. Show all intercepts. Confirm your work by writing your function in standard form $f(x) = a(x-h)^2 + k$ by completing the square, and using translations to graph.

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$$

$$f(-1) = 3 - 2(-1) - (-1)^2 = 3 + 2 - 1 = 4 \quad (-1, 4)$$

$$\begin{aligned} f(x) &= 3 - (x^2 + 2x) \\ &= 3 - (x^2+2x+1) + 1 \\ &= 4 - (x+1)^2 \end{aligned}$$

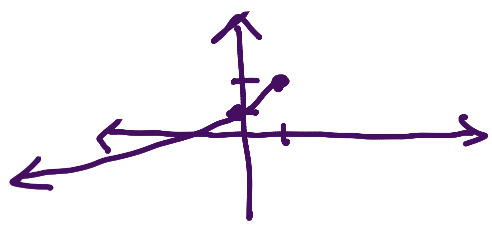


8. Sketch the graph of $f(x) = 2 - \sqrt{1-x}$. Starting with $y = \sqrt{x}$, list each translation used to graph $f(x)$.

$$y = -\sqrt{-(x-1)} + 2$$

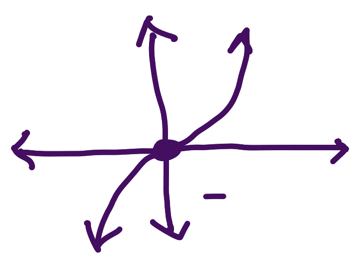
$\rightarrow 1$ reflect over x and y axis

$\uparrow 2$



9. Use the definition of absolute value to write the function $g(x) = x|x|$ as a piecewise defined function. Then sketch its graph.

$$g(x) = \begin{cases} x(-x) & x<0 \\ x(x) & x \geq 0 \end{cases} \qquad \begin{cases} -x^2 & x<0 \\ x^2 & x \geq 0 \end{cases}$$



10. Find the inverse of $f(x) = \sqrt{4-x}$. Be sure to include domain.

$$x = \sqrt{4-y}$$
$$x^2 = 4 - y$$
$$y = 4 - x^2$$
$$f^{-1}(x) = 4 - x^2$$

$x \geq 0$ $(0, \infty)$

11. Find the inverse of one-to-one function $f(x) = \dfrac{x+2}{x-3}$. Use that inverse function to find the range of $f(x)$. Then find the horizontal asymptote of $f(x)$ if possible.

$x = \dfrac{y+2}{y-3}$

$x(y-3) = y+2$

$xy - 3x = y+2$

$xy - y = 3x+2$

$y(x-1) = 3x+2$

$f^{-1}(x) = \dfrac{3x+2}{x-1}$

domain of f^{-1} is equal to the range of f $(-\infty, 1) \cup (1, \infty)$

horizontal asymptote $y=1$

12. Find the domain of the following functions:

(a) $f(x) = \sqrt{x^3 - x^2 - 6x}$ (b) $f(x) = \ln\left(\dfrac{8}{x} - 2\right)$

$x^3 - x^2 - 6x \geq 0$

$x(x^2 - x - 6) \geq 0$

$x(x-3)(x+2) \geq 0$

-2 $+$ 0 $-$ 3 $+$

$[-2, 0] \cup [3, \infty)$

$\dfrac{8}{x} - 2 > 0$

$\dfrac{8-2x}{x} > 0$

$-$ 0 $+$ 4 $-$

$(0, 4)$

13. Find the solution set of each of the following equations:

(a) $\log_3(2x^2 - 5) - \log_3 x = 1$ (b) $4^{3-x^2} = \left(\dfrac{1}{8}\right)^{x+1}$

(c) $\ln(x+8) + \ln(x-2) = \ln(3x+2)$

a) $\log_3\left(\dfrac{2x^2-5}{x}\right) = 1$

$3^1 = \dfrac{2x^2-5}{x}$

$0 = 2x^2 - 3x - 5$

$0 = 2x^2 - 5x + 2x - 5$

$= x(2x-5) + (2x-5)$

$(2x-5)(x+1)$

$x = 5/2, \cancel{-1}$

b) $(2^2)^{3-x^2} = (2^{-3})^{x+1}$

$2(3-x^2) = -3(x+1)$

$6 - 2x^2 = -3x - 3$

$0 = 2x^2 - 3x - 9$

$0 = (2x+3)(x-3)$

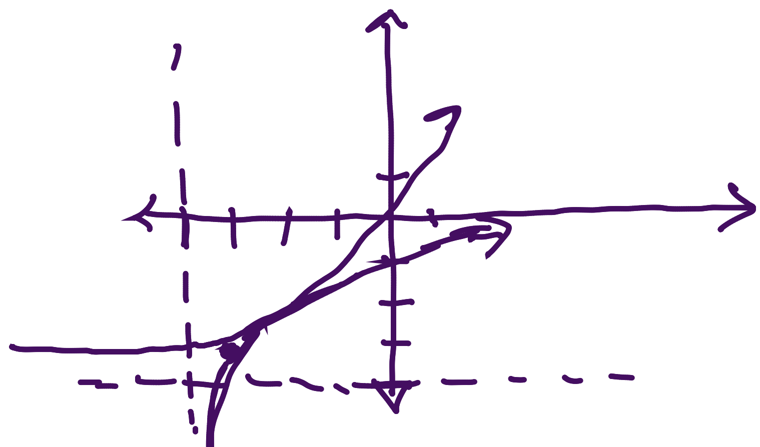
14. Find the inverse of $f(x) = e^{x+3} - 4$. Sketch the graph of f and f^{-1} on the same axes. Include at least one point and any asymptotes of each function.

$x = e^{y+3} - 4$

$x+4 = e^{y+3}$

$\ln(x+4) = y+3$

$f^{-1}(x) = \ln(x+4) - 3$



15. Let $f(x) = \begin{cases} x+4 & x < -2 \\ 2 - |x| & -2 \le x < 2 \\ \ln(x-1) & x > 2 \end{cases}$.

(a) Find if possible: $f(-4)$, $f(-2)$, $f(0)$, $f(2)$, $f(e+1)$.

(b) Sketch the graph of $y = f(x)$. (c) Use your graph to evaluate the following

a)

$f(-4) = -4+4 = 0$

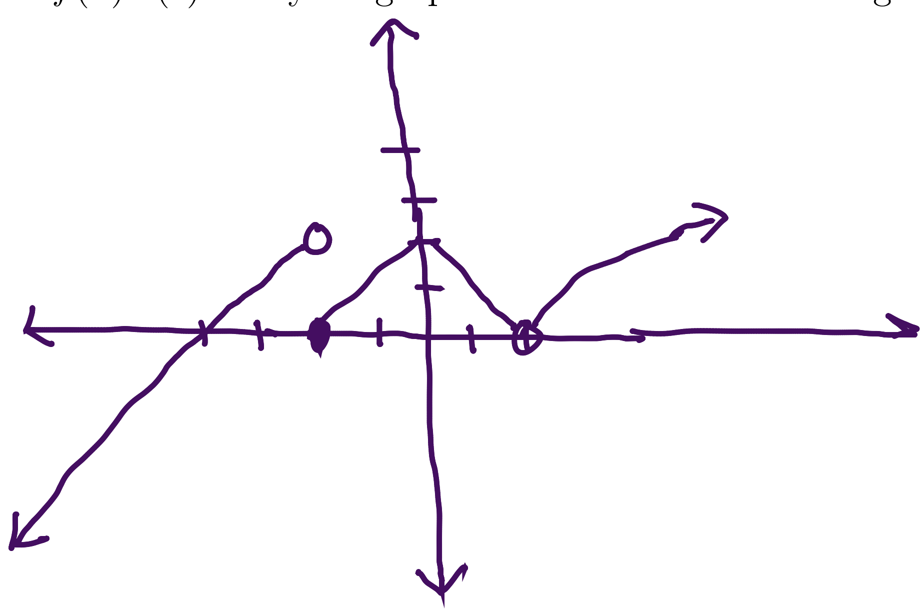
b) $f(-2) = 2 - |-2| = 0$

c) $f(0) = 2 - |0| = 2$

d) $f(2)$ is undefined

e) $f(e+1) = \ln(e+1-1)$

$= \ln(e) = 1$



limits if they exist:

1) $\lim_{x \to -2} f(x)$ 2) $\lim_{x \to 0} f(x)$ 3) $\lim_{x \to 2} f(x)$

$=$ DNE $= 2$ $= 0$

16. Let $f(x) = \dfrac{x^2 - 4}{x^2 - 2x - 8}$. Find:

(a) domain of f $(-\infty, -2) \cup (2, 4) \cup (4, \infty)$

(b) all intercepts (express as ordered pairs)

(c) all vertical and horizontal asymptotes

(d) Sketch the graph of $y = f(x)$. Include the coordinates of any holes in the function.

(e) Use your graph to find $\lim_{x \to -2} f(x)$.

$$f(x) = \frac{(x-2)(x+2)}{(x-4)(x+2)} = \frac{x-2}{x-4}$$

b) x-intercept

$$0 = \frac{x-2}{x+4}$$

$$0 = x-2$$

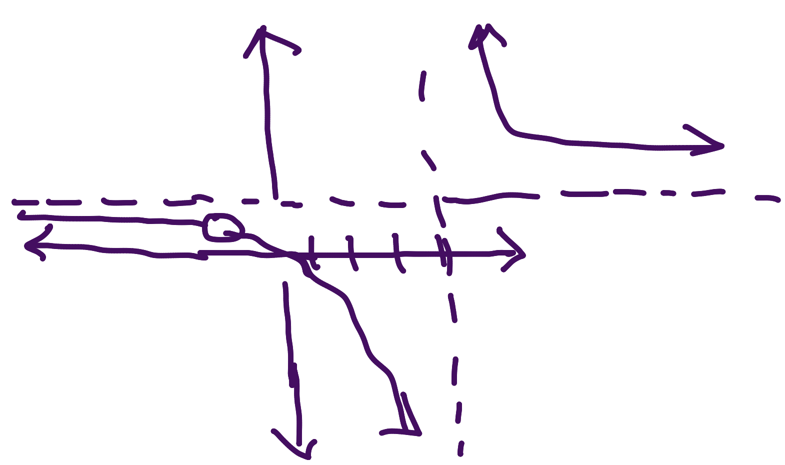
$x = 2$ $(2, 0)$

y-intercept

$$f(0) = \frac{0-2}{0-4} = \frac{1}{2} \quad (0, \tfrac{1}{2})$$

V.A $x = 4$

HA $y = 1$



$$\lim_{x \to -2} \frac{x-2}{x-4} = \frac{-2-2}{-2-4} = \frac{-4}{-6} = \frac{2}{3}$$

.1

17. There is a linear relationship between temperature in degrees Celsius C and degrees Fahrenheit F. Water freezes at $0°C$ ($32°F$) and boils at $100°C$ ($212°F$). Write the model expressing C as function of F. What is the temperature in degrees Fahrenheit if the temperature is $30°C$? What does the slope of the line tell you?

$(32,0), (212,100)$

$m = \frac{0-100}{32-212} = \frac{-100}{-180} = \frac{5}{9}$

$C = mF + b$

$0 = \frac{5}{9}(32) + b$

$b = -\frac{160}{9}$

$C = \frac{5}{9}F - \frac{160}{9}$

tempature increases 5°C as Fahrenheit temp increases by 9°

18. The demand and supply functions for a given product are given by

$p = D(q) = 60 - 2q^2$ and $p = S(q) = q^2 + 9q + 30$ where q is quantity in thousands and p is the unit price. Find the equilibrium quantity and price.

How many items will the supplier provide if the unit price of the product is \$40? What will be the demand for the product when the unit price is \$40? What should happen to the price of the product?

$60 - 2q^2 = q^2 + 9q + 30$

$0 = 3q^2 + 9q - 30$

$0 = 3(q^2 + 3q - 10)$

$= 3(q+5)(q-2)$ $\quad q = 2$

$p = D(2) = 60 - 2(2)^2 = 52$ $\quad$ \$52

$40 = q^2 + 9q + 30$

$0 = q^2 + 9q - 10$

$= (q+10)(q-1)$ $\quad q = 1$ $\quad$ (1000 items)

$40 = 60 - 2q^2$ $\quad 2q^2 = -20$

$q = \sqrt{10}$

price will rise

19. A financial manager at Target has made the following observations about a certain product in one of its districts: an average of 250 units will sell in a month when the price is \$15, but an average of 50 more will sell if the price is reduced by \$1. Assuming the demand function is linear,

(a) Express p as a function of x.

(b) Find the revenue function $R(x)$. Find the production level x that will maximize revenue. What is the maximum revenue?

(c) If fixed costs are \$800 and the marginal cost is \$10 per item, find each value of x at which the company will break even. What is the profit for those values?

(d) Find the profit function $P(x)$. What price should the manager charge to maximize profit on this item?

a) $m = \dfrac{15-14}{250-300} = -\dfrac{1}{50}$

$15 = -\dfrac{1}{50}(250) + b$

$15 = -5 + b \quad b = 20$

$p = -\dfrac{1}{50}x + 20$

b) $R(x) = xp = x\left(-\dfrac{1}{50}x + 20\right)$

$= -\dfrac{1}{50}x^2 + 20x$

$x = -\dfrac{b}{2a} = \dfrac{-20}{2\left(-\frac{1}{50}\right)} = \dfrac{-20}{-\frac{1}{25}} = 500$

$R(500) = -\dfrac{1}{50}(500)^2 + 20(500)$

$= -5000 + 10{,}000 = 5000 \leftarrow$ maximum revenue

c.) revenue = cost

$-\dfrac{1}{50}x^2 + 20x = 10x + 800$

$0 = \dfrac{1}{50}x^2 - 10x + 800$

$= x^2 - 500x + 40{,}000$

$(x - 400)(x - 100)$

$x = 400, 100$

profit is 0

d) Profit = Revenue − Cost

$$P(x) = -\frac{1}{50}x^2 + 20x - (10x + 800)$$

$$= -\frac{1}{50}x^2 + 10x - 800$$

$$x = \frac{-b}{2a}$$

$$= \frac{-10}{2\left(-\frac{1}{50}\right)}$$

$$= 250$$

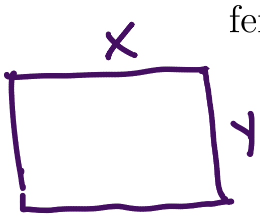
$$p = -\frac{1}{50}x + 20$$

$$p = -\frac{1}{50}(250) + 20$$

$$= -5 + 20$$

$$= 15$$

20. A farmer plans to spend \$6000 to enclose a rectangular field with two kinds of fencing. Two opposite sides will require heavy-duty fencing that costs \$3 per linear foot, while the other two sides can be constructed with standard fencing that costs \$2 per foot. Express the area of the field, A, as a function of x, the length of a side that requires the more expensive fence. Find the value of x that will maximize the area of the field, and the length of a side that uses standard fencing.



$A = xy$

$2(3x) + 2(2y) = 6000$

$3x + 2y = 3000$

$2y = 3000 - 3x$

$y = 1500 - \frac{3}{2}x$

$A(x) = x\left(1500 - \frac{3}{2}x\right)$

$= 1500x - \frac{3}{2}x^2$

$x = -\frac{b}{2a} = \frac{-1500}{2\left(-\frac{3}{2}\right)} = 500$

$y = 1500 - \frac{3}{2}(500)$

$= 750$

21. Rewrite the expression as the sum, difference, or multiple of logarithms:

(a) $\log \dfrac{x^2}{1000}$ (b) $\ln \sqrt[3]{\dfrac{e^{x+1}(x-2)^4}{x^6}}$

(a) $= \log x^2 - \log 1000$

$= 2\log x - 3$

(b) $\ln \sqrt[3]{\dfrac{e^{x+1}(x-2)^4}{x^6}} = \frac{1}{3}\ln\left(\dfrac{e^{x+1}(x-2)^4}{x^6}\right)$

$= \frac{1}{3}\left[\ln e^{x+1} + \ln(x-2)^4 - \ln x^6\right)$

$= \frac{1}{3}\left(x+1 + 4\ln(x-2) - 6\ln x\right)$

22. Mr. Jones invested \$2500 at 5.5% compounded continuously. How long will it take his account to grow to \$4000 if he adds no new funds to the account?

$$\frac{4000}{2500} = \frac{2500}{2500} e^{.055t} \qquad t = \frac{\ln(8/5)}{.055} \approx 8.5 \text{ years}$$

$$\frac{8}{5} = e^{.055t}$$

$$\ln\left(\frac{8}{5}\right) = .055t$$

23. How much money must be invested now at 3 1/4% compounded quarterly in order to have \$6000 in three years?

$$6000 = P_0\left(1 + \frac{.0325}{4}\right)^{4 \cdot 3}$$

$$P_0 = \frac{6000}{\left(1 + \frac{.0325}{4}\right)^{12}} \approx \$5444.76$$

24. Iodine - 131 has a half-life of 8 days. Suppose some hay was contaminated with ten times the allowable amount of I-131. How long must the hay be stored before it can be fed to cattle? Hint: the hay must have one-tenth of its current amount of I-131.

$$\frac{1}{2}\cancel{Q_0} = \cancel{Q_0} e^{k(8)}$$

$$\frac{1}{2} = e^{8k}$$

$$\ln\left(\frac{1}{2}\right) = 8k$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{8}$$

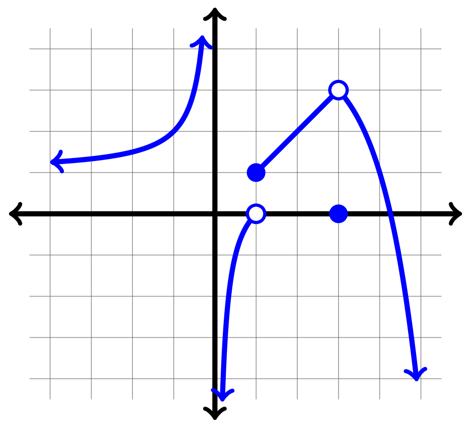
$$Q(t) = Q_0 e^{\frac{\ln\left(\frac{1}{2}\right)}{8} t}$$

$$\frac{1}{10}\cancel{Q_0} = \cancel{Q_0} e^{\frac{\ln\left(\frac{1}{2}\right)}{8} t}$$

$$\ln\left(\frac{1}{10}\right) = \frac{\ln\left(\frac{1}{2}\right)}{8} t$$

$$t = \frac{8\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)} \approx 26.57$$

25. Use the following graph of a function $f(x)$ to evaluate the limits and function value if possible. If the limit does not exist, write "dne".



a) $\lim_{x\to 0^-} f(x) = \infty$ b) $\lim_{x\to 0^+} f(x)$ $-\infty$ c) $\lim_{x\to 0} f(x) =$ DNE

d) $\lim_{x\to 1+} f(x) = 1$ e) $\lim_{x\to 1^-} f(x) = 0$ f) $\lim_{x\to 1} f(x) =$ DNE

g) $\lim_{x\to 3} f(x) = 3$ h) $f(3) = 0$ i) $\lim_{x\to -1} f(x) = 2$

26. Use the properties of limits to evaluate $\lim_{x\to a} \frac{(fg)(x)}{\sqrt[3]{g(x)-1}}$ if $\lim_{x\to a} f(x) = -\frac{1}{3}$ and $\lim_{x\to a} g(x) = 9$.

$$\lim_{x\to a} \frac{f(x)\,g(x)}{\sqrt[3]{g(x)-1}} = \frac{-\frac{1}{3}(9)}{\sqrt[3]{9-1}} = \frac{-3}{\sqrt[3]{8}} = -\frac{3}{2}$$

27. Evaluate (a) $\lim_{x\to -1} \frac{x+\sqrt{x+2}}{x+1}$ and (b) $\lim_{x\to 2} \frac{\frac{2}{x}-1}{x-2}$.

a) $\lim_{x\to -1} \frac{x+\sqrt{x+2}}{x+1} \cdot \frac{x-\sqrt{x+2}}{x-\sqrt{x+2}} = \lim_{x\to -1} \frac{x^2-(x+2)}{(x+1)(x-\sqrt{x+2})}$

$= \lim_{x\to -1} \frac{(x-2)\cancel{(x+1)}}{\cancel{(x+1)}(x-\sqrt{x+2})}$

$= \lim_{x\to -1} \frac{x-2}{x-\sqrt{x+2}}$

$= \frac{-1-2}{-1-\sqrt{-1+2}} = \frac{-3}{-2} = \frac{3}{2}$

(b) $\lim_{x\to 2} \frac{\frac{2}{x}-1}{x-2} \cdot \frac{x}{x} = \lim_{x\to 2} \frac{2-x}{x(x-2)}$

$= \lim_{x\to 2} \frac{-\cancel{(-2+x)}}{x\cancel{(x-2)}}$

$= \lim_{x\to 2} -\frac{1}{x} = -\frac{1}{2}$

28. If $f(x) = \begin{cases} \dfrac{x^2-16}{x^2+3x-4} & x \neq -4 \\ 0 & x = -4 \end{cases}$

find $p = \lim_{x\to -4} f(x)$ and $q = \lim_{x\to 1^-} f(x)$.

$\frac{x^2-16}{x^2+3x-4} = \frac{(x-4)(x+4)}{(x+4)(x-1)} = \frac{x-4}{x-1}$

$\lim_{x\to -4} \frac{x-4}{x-1} = \frac{-8}{-5}$

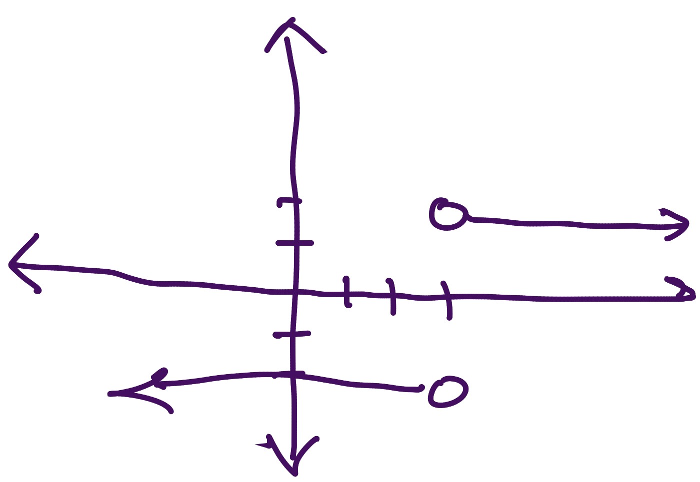
$\lim_{x\to 1^-} \frac{x-4}{x-1}$ $\frac{\text{approaches } -5}{\text{approaches 0 and negative}} = -\infty$

29. Sketch the graph of $f(x) = \dfrac{|6-2x|}{x-3}$. Hint: rewrite as a piecewise function without absolute value bars.

Use the graph to find: (a) $\lim_{x\to 3^-} f(x)$, (b) $\lim_{x\to 3^+} f(x)$, and (c) $\lim_{x\to 3} f(x)$.

Now find those limits algebraically without using the graph.

$$\begin{cases} \dfrac{-(6-2x)}{x-3} & 6-2x<0 \\ \dfrac{6-2x}{x-3} & 6-2x>0 \end{cases} \qquad \begin{cases} 2 & x>3 \\ -2 & x<3 \end{cases}$$



$$\lim_{x\to 3^-} -2 = -2$$

$$\lim_{x\to 3^+} 2 = 2$$

30. If $f(x) = \dfrac{x^3+3x^2+2x}{x-x^3}$, find a) $\lim_{x\to 0^+} f(x)$ b) $\lim_{x\to -1^+} f(x)$, c) $\lim_{x\to 1^-} f(x)$

and d) $\lim_{x\to -\infty} f(x)$. Find each vertical and horizontal asymptote of $f(x)$.

$$f(x) = \frac{x^3+3x^2+2x}{x-x^3} = \frac{x(x^2+3x+2)}{x(1-x^2)}$$

$$= \frac{(x+2)(x+1)}{(1-x)(1+x)}$$

$$= \frac{x+2}{1-x}$$

a) $\lim_{x\to 0^+} f(x) = 2$

b) $\lim_{x\to -1^+} f(x) = \frac{1}{2}$

c) $\lim_{x\to 1^-} f(x) = \infty$

d) $\lim_{x\to -\infty} \dfrac{1+\cancel{\frac{2}{x}}^{\to 0}}{\cancel{\frac{1}{x}}^{\to 0}-1} = \dfrac{1}{-1} = -1$

31. If $f(x) = \dfrac{2}{e^{-x} - 3}$, find if possible:

1) $\lim_{x \to -\infty} f(x)$ 2) $\lim_{x \to +\infty} f(x)$ 3) Each asymptote of the graph of $f(x)$.

$$\lim_{x \to -\infty} \frac{2}{\frac{1}{e^x} - 3} = \lim_{x \to -\infty} \frac{2\cancel{e^x}^{0}}{1 - 3\cancel{e^x}^{0}} = 0$$

$$\lim_{x \to \infty} \frac{2}{\cancel{\frac{1}{e^x}}^{0} - 3} = -\frac{2}{3}$$

$$e^{-x} = 3$$
$$\frac{1}{3} = e^x$$
$$\ln\left(\frac{1}{3}\right) = x$$

asymptotes

$y = 0$

$y = -\frac{2}{3}$

$x = \ln\left(\frac{1}{3}\right)$

32. The Intermediate Value Theorem guarantees that the function $f(x) = x^3 - \dfrac{1}{x} - 5x + 3$ has a zero on which of the following intervals?

a) $[-1, 1]$ b) $[1, 3]$ c) $[3, 5]$ d) $[-3, -2]$

b)

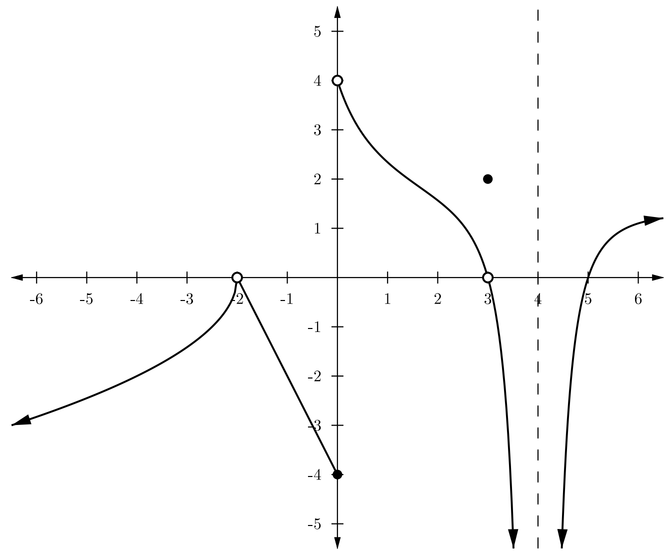
$$\left.\begin{array}{l} f(1) = (1)^3 - \frac{1}{1} - 5(1) + 3 = -2 < 0 \\ f(3) = 3^3 - \frac{1}{3} - 5(3) + 3 > 0 \end{array}\right\} \text{changes signs}$$

d)

$$\left.\begin{array}{l} f(-3) = (-3)^3 - \frac{1}{-3} - 5(-3) + 3 < 0 \\ f(-2) = (-2)^3 - \frac{1}{-2} - 5(-2) + 3 \end{array}\right\} \text{changes signs}$$

$$8 + \frac{1}{2} \qquad 10 + 3 > 0$$

33. Consider a function $f(x)$ which has the following graph.



(a) On which interval(s) is $f(x)$ continuous? $(-\infty,-2),(-2,0),(0,3),(3,4),(4,\infty)$

(b) $f(x)$ has a jump discontinuity at $x =$ <u>0</u>.

(c) $f(x)$ has an infinite discontinuity at $x =$ <u>4</u>.

(d) $f(x)$ has a removable discontinuity at $x =$ <u>-2, 3</u>.

(e) How would you define or redefine $f(x)$ at the point(s) in part (d) in order to make $f(x)$ continuous?

$f(-2) = 0$ $f(3) = 0$

MAC 2233: Unit 2 Exam Review
Lectures 15 – 24

1. Use the definition of derivative to evaluate $f'(x)$ if $f(x) = \sqrt{2x-1}$. Check your answer using a derivative rule.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2(x+h)-1} + \sqrt{2x-1}}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}$$

$$= \lim_{h\to 0} \frac{2(x+h)-1 - (2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h \cancel{-1} \cancel{-2x} \cancel{+1}}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)} = \frac{2}{\sqrt{2x-1} + \sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$$

2. (a) Use the **definition of derivative** to find $f'(x)$ if $f(x) = \dfrac{x}{2x-1}$. Check your answer using the Quotient Rule.

(b) Find each interval over which $f(x)$ is differentiable.

(c) Write the equation of the tangent line to $f(x) = \dfrac{x}{2x-1}$ at $x = -1$.

$$\lim_{h\to 0} \frac{\dfrac{x+h}{2(x+h)-1} - \dfrac{x}{2x-1}}{h} \cdot \frac{(2(x+h)-1)(2x-1)}{(2(x+h)-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{(2x-1)(x+h) - x(2x+2h-1)}{h(2x+2h-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2} + \cancel{2xh} \cancel{-x} - h - \cancel{2x^2} - \cancel{2xh} \cancel{+x}}{h(2x+2h-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(2x+2h-1)(2x-1)} = \frac{-1}{(2x-1)^2}$$

b) $f(x) = \dfrac{x}{2x-1}$

$$f'(x) = \frac{1(2x-1) - x(2)}{(2x-1)^2}$$

$$= \frac{2x - 1 - 2x}{(2x-1)^2}$$

$$= \frac{-1}{(2x-1)^2}$$

---

c) $x \neq \frac{1}{2}$

$\left(-\infty, \frac{1}{2}\right), \left(\frac{1}{2}, \infty\right)$

3. Indicate whether each of the following statements is true or false.
   (a) If f is continuous at $x = a$, then f is differentiable at $x = a$. False $f(x)=|x|$ at $x=a$
   (b) If f is not continuous at $x = a$, then f is not differentiable at $x = a$.
   (c) If f has a vertical tangent line at $x = a$, then the graph of $f'(x)$ has a vertical asymptote at $x = a$.

true

4. If an object is projected upward from the roof of a 200 foot building at 64 ft/sec, its height h in feet above the ground after t seconds is given by
   $h(t) = 200 + 64t - 16t^2$. Find the following:
   (a) The average velocity of the object from time $t = 0$ until it reaches its maximum height (hint: consider the graph of the function)
   (b) The instantaneous velocity of the object at time $t = 1$ second using the limit definition.

a) reaches maximum height at vertex

$$t = -\frac{b}{2a} = \frac{-64}{2(-16)} = 2$$

average velocity from $t=0$ to $t=2$

$$\frac{h(2)-h(0)}{2-0} = \frac{\cancel{200} + 64(2) - 16(2)^2 - (\cancel{200} + 64(0) - 16 \cdot 0^2)}{2}$$

$$= \frac{128-64}{2}$$

$$= 32 \text{ ft/sec}$$

b) $h'(t) = 64 - 32t$ or $\lim\limits_{t \to 1} \frac{h(t)-h(1)}{t-1}$

$h'(1) = 64 - 32 = 32$

$$2y = 8x - 9$$

$$y = 4x - \frac{9}{2} \qquad m = 4$$

5. Find each value at which $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x$ is parallel to the line $2y - 8x + 9 = 0$.

$$f'(x) = x^2 - x - 2 = 4$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, -2$$

$$y = x^2 - 2x^{\frac{1}{2}} + 1$$

6. Find the value of a so that the tangent line to $y = x^2 - 2\sqrt{x} + 1$ is perpendicular to the line $ay + 2x = 2$ when $x = 4$.

$$y' = 2x - 2\left(\tfrac{1}{2}\right)x^{-\frac{1}{2}} = 2x - x^{-\frac{1}{2}}$$

$$ay + 2x = 2$$

$$ay = -2x + 2$$

$$y = -\frac{2}{a}x + \frac{2}{a} \qquad m_\perp = \frac{a}{2}$$

$$y'\Big|_4 = 2(4) - \frac{1}{\sqrt{4}} = 8 - \frac{1}{2} = \frac{15}{2}$$

$$\frac{a}{2} = \frac{15}{2}$$

$$a = 15$$

$f(x) = 2x^3\sqrt{x} + x^3 - 4x\sqrt{x} - 2x$

$= 2x^{7/2} + x^3 - 4x^{3/2} - 2x$

7. If $f(x) = (x^3 - 2x)(2\sqrt{x} + 1)$, find $f'(x)$ two ways: rewriting $f(x)$ and differentiating, and using the Product Rule.

$f'(x) = 7x^{5/2} + 3x^2 - 6x^{1/2} - 2$

product rule

$f'(x) = (3x^2 - 2)(2\sqrt{x} + 1) + (x^3 - 2x)(x^{-\frac{1}{2}})$

8. Find each value of x at which $f(x) = (1 - x)^5(5x + 2)^4$ has a horizontal tangent line.

$f'(x) = -5(1-x)^4(5x+2)^4 + 4(1-x)^5(5x+2)^3(5)$

$= -5(1-x)^4(5x+2)^3(5x+2 - 4(1-x))$

$= -5(1-x)^4(5x+2)^3(5x+2-4+4x)$

$0 = -5(1-x)^4(5x+2)(9x+2)$

$x = 1, -\frac{2}{5}, -\frac{2}{9}$

9. Let $f(x) = \dfrac{(\sqrt{x} - 1)^2}{x}$. Find $f'(x)$ and write as a single fraction. Write the equation of the tangent line to $f(x)$ at $x = 4$.

$f(x) = \dfrac{(\sqrt{x}-1)(\sqrt{x}-1)}{x} = \dfrac{x - 2\sqrt{x} + 1}{x} = 1 - 2x^{-\frac{1}{2}} + x^{-1}$

$f'(x) = x^{-3/2} - x^{-2}$

$= \dfrac{1}{x^{3/2}} - \dfrac{1}{x^2}$

$= \dfrac{x^{1/2}}{x^2} - \dfrac{1}{x^2} = \dfrac{\sqrt{x}-1}{x^2}$

$f'(4) = \dfrac{\sqrt{4}-1}{4^2} = \dfrac{1}{16}$

$f(4) = \dfrac{(\sqrt{4}-1)^2}{4} = \dfrac{1}{4}$

$\frac{1}{4} = \frac{1}{16}(4) + b \quad b = 0$

$\boxed{y = \frac{1}{16}x}$

$f(3) = 1$

10. Write the equation of the tangent line to $f(x) = \left(x - \frac{6}{x}\right)^3$ at $x = 3$.

$f'(x) = 3\left(x - \frac{6}{x}\right)^2\left(1 + \frac{6}{x^2}\right)$ $\quad f'(3) = 3\left(3 - \frac{6}{3}\right)^2\left(1 + \frac{6}{9}\right)$

$= 3(1)\left(\frac{15}{9}\right) = 5$

$y = mx + b$

$1 = 5(3) + b \Rightarrow b = -14$

$y = 5x - 14$

11. Find each value of x at which the function $f(x) = \dfrac{\sqrt[3]{6x+1}}{x}$ has (a) horizontal and (b) vertical tangent lines. Write the equation of each of those lines.

$f(x) = \dfrac{(6x+1)^{1/3}}{x}$ $\quad f'(x) = \dfrac{\frac{1}{3}(6x+1)^{-\frac{2}{3}}6x - (6x+1)^{1/3}}{x^2}$

$= \dfrac{\dfrac{2x}{(6x+1)^{2/3}} - (6x+1)^{1/3}}{x^2}$

$= \dfrac{2x - (6x+1)}{(6x+1)^{2/3}x^2} = \dfrac{-4x - 1}{(6x+1)^{2/3}x^2}$

horizontal tangent line at $x = -\frac{1}{4}$ $\quad f\left(\frac{1}{4}\right) = 2^{5/3}$

$y = 2^{5/3}$

vertical tangent line $\quad x = -\frac{1}{6}, 0$

12. Suppose that $f(4) = -1$, $g(4) = 2$, $f(-4) = 1$, $g(-4) = 3$, $f'(4) = -2$, $g'(4) = 12$, $f'(-4) = 6$, and $g'(-1) = -2$.

Find: (a) $h'(4)$ if $h(x) = g(f(x))$ and (b) $H'(4)$ if $H(x) = \sqrt{xf(x) + \frac{x^2}{2}}$.

a) $h'(x) = g'(f(x))\, f'(x)$

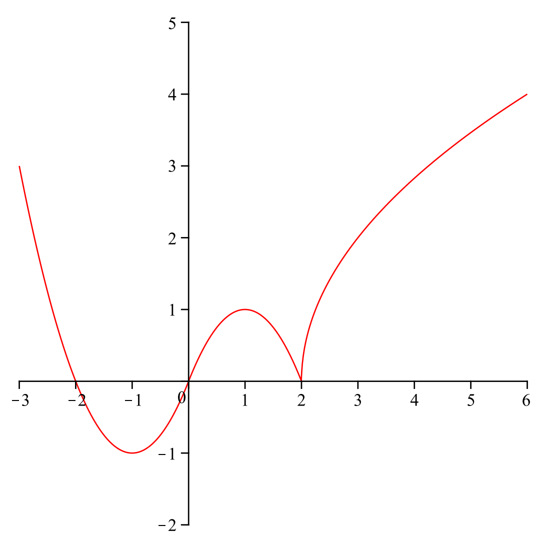
$h'(4) = g'(f(4))\, f'(4) = g'(-1)(-2) = -2(-2) = 4$

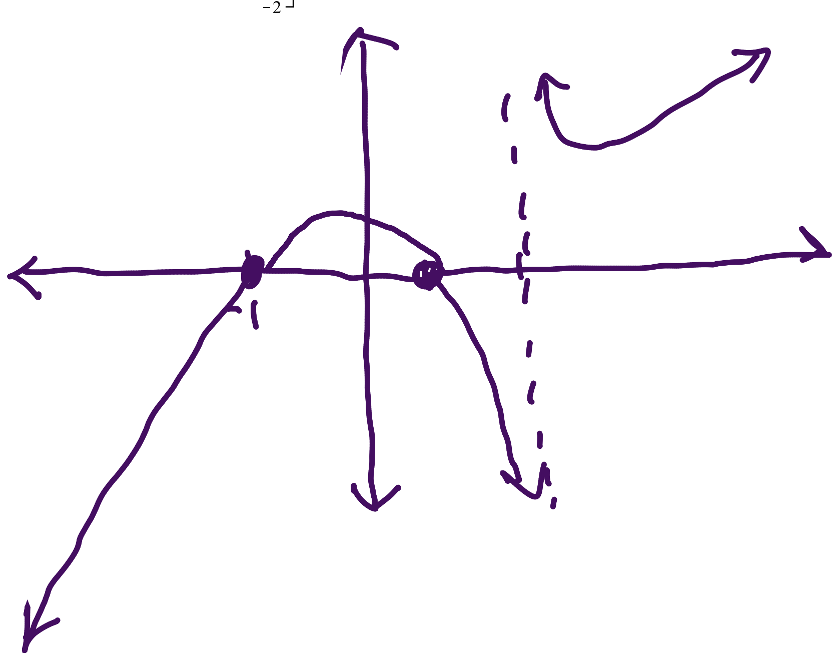
b) $H'(x) = \frac{1}{2}\left(x\, f(x) + \frac{x^2}{2}\right)^{-\frac{1}{2}} \left(f(x) + x f'(x) + x\right)$

$H'(4) = \frac{1}{2}\left(4\, f(4) + \frac{16}{2}\right)^{-\frac{1}{2}} \left(f(4) + 4 f'(4) + 4\right)$

$= \frac{1}{2} \frac{1}{(4(-1)+8)^{1/2}} \left(-1 + 4(-2) + 4\right) = \frac{1}{2} \cdot \frac{1}{2}(-5) = -\frac{5}{4}$

13. Sketch a possible graph of the derivative of the function $y = f(x)$ shown below.





14. Find the derivative:

(a) $f(x) = 3^{2x-1}$ (b) $f(x) = \log_4 (x^2 - x)$

a)

$$f'(x) = 2\ln(3)\, 3^{2x-1}$$

b) $$f'(x) = \frac{2x-1}{\ln(4)(x^2-x)}$$

15. Find the slope of the tangent line to the curve given by $\sqrt{3x-y} - e^{x+y} = 1 + \ln x$ at $(1, -1)$.

$$\frac{1}{2}(3x-y)^{-\frac{1}{2}}\left(3-\frac{dy}{dx}\right) - e^{x+y}\left(1+\frac{dy}{dx}\right) = \frac{1}{x}$$

$$\frac{1}{2}(3-(-1))^{-\frac{1}{2}}(3-m) - e^{0}(1+m) = 1$$

$$\frac{1}{4}(3-m) - (1+m) = 1$$

$$\frac{3}{4} - \frac{1}{4}m - 1 - m = 1$$

$$-\frac{5}{4} = \frac{5}{4}m$$

$$m = -1$$

16. Find the first three derivatives of $f(x) = \ln(x^2+1)$.

$$f'(x) = \frac{2x}{x^2+1}$$

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

$$f^{(3)}(x) = \frac{-4x(x^2+1)^{\cancel{2}} - 2(2-2x^2)\cancel{(x^2+1)}(2x)}{(x^2+1)^{\cancel{4}}}$$

$$= \frac{-4x(x^2+1) - 4x(2-2x^2)}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3}$$

$f(x) = \ln(2x-6) - \frac{1}{2}\ln(x^2+3)$

17. Find each value of x at which $f(x) = \ln\left(\frac{2x-6}{\sqrt{x^2+3}}\right)$ has a horizontal tangent line.

$$f'(x) = \frac{2}{2x-6} - \frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+3)}$$

$$0 = \frac{1}{x-3} - \frac{x}{x^2+3}$$

$$0 = x^2+3 - (x-3)x$$

$$0 = x^2+3 - x^2+3x$$

$$0 = 3+3x$$

$$x = -1$$

-1 is not in the domain of f

no horizontal tangent lines

18. Find $f'(x)$ if $f(x) = \ln \frac{e^{x-3}\sqrt[3]{6+3x}}{(3x+1)^2}$.

$$f(x) = \ln e^{x-3} + \frac{1}{3}\ln(6+3x) - 2\ln(3x+1)$$

$$= 1 + \frac{3}{3(6+3x)} - \frac{2(3)}{3x+1}$$

$$= 1 + \frac{1}{6+3x} - \frac{6}{3x+1}$$

19. Use Logarithmic Differentiation to find the slope of the tangent line to $f(x) = x^{\sqrt{x}}$ at $x = 4$.

$$y = x^{\sqrt{x}}$$

$$\ln y = \ln x^{\sqrt{x}}$$

$$= \sqrt{x}\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\ln x + \frac{\sqrt{x}}{x}$$

$$\frac{dy}{dx} = y\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$= x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$f'(4) = 4^2\left(\frac{\ln(4)}{2\cdot 2} + \frac{1}{2}\right)$$

$$= 16\left(\frac{\ln 4}{4} + \frac{1}{2}\right)$$

$$= 4\ln 4 + 8$$

- next exam

20. Find each value at which $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2$ has a relative maximum or minimum.
Find the absolute extrema of $f(x)$ on $[-2, 1]$.

21. Let $f(x) = \frac{\sqrt[3]{3x-2}}{x}$.

(a) Find $f'(x)$ and write as a single fraction.
(b) Find the equation of each horizontal and vertical tangent line of $f(x)$.
(c) Find each x-value at which $f(x)$ has a critical number.
(d) ~~Find the relative extreme values of $f(x)$~~.

$$f'(x) = \frac{\frac{1}{\cancel{3}}(3x-2)^{-\frac{2}{3}}(\cancel{3})\,x - (3x-2)^{\frac{1}{3}}}{x^2}$$

$$= \frac{\frac{x}{(3x-2)^{2/3}} - (3x-2)^{1/3}}{x^2}$$

$$= \frac{x - (3x-2)}{x^2(3x-2)^{2/3}}$$

$$= \frac{-2x+2}{x^2(3x-2)^{2/3}}$$

horizontal tangent line at $x=1$ $f(1)=1$ $y=1$

vertical tangent line $x=2/3$

critical numbers at $x=1, 2/3$

22. The cost function for a product is
$C(x) = 1.25x^2 + 25x + 8000$.

(a) Suppose the company decides to increase production by 4 units per day when the current daily production level is 50 units. Find the rate of change of cost with respect to time.

(b) If $C(x) = 1.25x^2 + 25x + 8000$, find each interval on which **average cost** is increasing and decreasing. For what production level x is average cost minimized?

a)

$\frac{dx}{dt} = 4 \qquad \frac{dC}{dt} = ?$ when $x = 50$

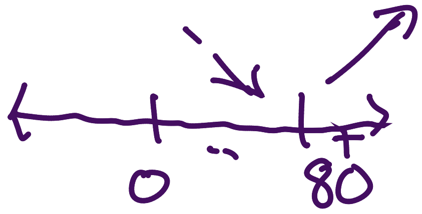
$$\frac{dC}{dt} = \frac{5}{2}x\frac{dx}{dt} + 25\frac{dx}{dt}$$

$$\frac{dC}{dt} = \frac{5}{2}(50)(4) + 25(4) = 500 + 100 = \$600 \text{ / day}$$

b) $\bar{C}(x) = \dfrac{1.25x^2 + 25x + 8000}{x} = 1.25x + 25 + \dfrac{8000}{x}$

$$\bar{C}'(x) = 1.25 - \frac{8000}{x^2} = 0$$

$$1.25x^2 = 8000$$
$$x^2 = 6400$$
$$x = 80$$



decreasing on $(0, 80)$
increasing for $x > 80$

$p(7500) = -.02(7500) + 400$

at \$250

23. The demand function for a certain product is given by

$p(x) = -0.02x + 400$, $0 \leq x \leq 20,000$, where p is the unit price when x items are sold. The cost function for the product is

$C(x) = 100x + 300,000$.

(a) Find the marginal profit of the product when $x = 2000$.

(b) Find the actual profit from the sale of the 2001st item. Compare to your answer in (a).

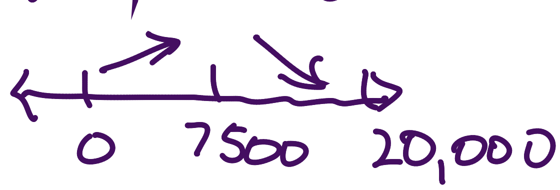
(c) Find each interval on which the profit function $P(x) = -0.02x^2 + 300x - 300,000$ is increasing and decreasing. Remember that $0 \leq x \leq 20,000$. How many items should be sold to maximize profit? At what price?

a) Profit = revenve - Cost

$$P(x) = xp(x) - C(x)$$
$$= x(-.02x + 400) - (100x + 300,000)$$
$$= -.02x^2 + 400x - 100x - 300,000$$
$$= -.02x^2 + 300x - 300,000$$
$$P'(x) = -.04x + 300$$
$$P'(2000) = -.04(2000) + 300 = 220$$

b) $\Delta P = P(2001) - P(2000) = 219.98$

c) $0 = -.04x + 300$ $\quad x = 7500$



increasing $(0, 7500)$ decreasing $(7500, 20,000)$

Profit is maximized when 7500 are sold