MAC 2233: Unit 2 Exam Review
Lectures 15 – 24

1. Use the definition of derivative to evaluate $f'(x)$ if $f(x) = \sqrt{2x-1}$. Check your answer using a derivative rule.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \cdot \frac{\sqrt{2(x+h)-1} + \sqrt{2x-1}}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}$$

$$= \lim_{h\to 0} \frac{2(x+h)-1 - (2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h \cancel{-1} \cancel{-2x} \cancel{+1}}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)} = \frac{2}{\sqrt{2x-1}+\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$$

2. (a) Use the **definition of derivative** to find $f'(x)$ if $f(x) = \dfrac{x}{2x-1}$. Check your answer using the Quotient Rule.

   (b) Find each interval over which $f(x)$ is differentiable.

   (c) Write the equation of the tangent line to $f(x) = \dfrac{x}{2x-1}$ at $x = -1$.

$$\lim_{h\to 0} \frac{\dfrac{x+h}{2(x+h)-1} - \dfrac{x}{2x-1}}{h} \cdot \frac{(2(x+h)-1)(2x-1)}{(2(x+h)-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{(2x-1)(x+h) - x(2x+2h-1)}{h(2x+2h-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2} + \cancel{2xh} \cancel{-x} - h - \cancel{2x^2} - \cancel{2xh} + \cancel{x}}{h(2x+2h-1)(2x-1)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(2x+2h-1)(2x-1)} = \frac{-1}{(2x-1)^2}$$

b) $f(x) = \dfrac{x}{2x-1}$

$$f'(x) = \frac{1(2x-1) - x(2)}{(2x-1)^2}$$

$$= \frac{2x - 1 - 2x}{(2x-1)^2}$$

$$= \frac{-1}{(2x-1)^2}$$

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c) $x \neq \dfrac{1}{2}$

$(-\infty, \tfrac{1}{2}), (\tfrac{1}{2}, \infty)$

3. Indicate whether each of the following statements is true or false.

   (a) If f is continuous at $x = a$, then f is differentiable at $x = a$. False $f(x) = |x|$ at $x = a$

   (b) If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

   (c) If f has a vertical tangent line at $x = a$, then the graph of $f'(x)$ has a vertical asymptote at $x = a$.

true

4. If an object is projected upward from the roof of a 200 foot building at 64 ft/sec, its height h in feet above the ground after t seconds is given by $h(t) = 200 + 64t - 16t^2$. Find the following:

   (a) The average velocity of the object from time $t = 0$ until it reaches its maximum height (hint: consider the graph of the function)

   (b) The instantaneous velocity of the object at time $t = 1$ second using the limit definition.

a) reaches maximum height at vertex

$$t = \frac{-b}{2a} = \frac{-64}{2(-16)} = 2$$

average velocity from $t=0$ to $t=2$

$$\frac{h(2)-h(0)}{2-0} = \frac{\cancel{200} + 64(2) - 16(2)^2 - (\cancel{200} + 64(0) - 16 \cdot 0^2)}{2}$$

$$= \frac{128 - 64}{2}$$

$$= 32 \text{ ft/sec}$$

b) $h'(t) = 64 - 32t$ or $\lim_{t \to 1} \frac{h(t) - h(1)}{t-1}$

$h'(1) = 64 - 32 = 32$

$$2y = 8x - 9$$
$$y = 4x - \frac{9}{2} \qquad m = 4$$

5. Find each value at which $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x$ is parallel to the line $2y - 8x + 9 = 0$.

$$f'(x) = x^2 - x - 2 \quad = 4$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3, -2$$

$$y = x^2 - 2x^{\frac{1}{2}} + 1$$

6. Find the value of a so that the tangent line to $y = x^2 - 2\sqrt{x} + 1$ is perpendicular to the line $ay + 2x = 2$ when $x = 4$.

$$y' = 2x - 2\left(\tfrac{1}{2}\right)x^{-\frac{1}{2}} = 2x - x^{-\frac{1}{2}}$$

$$ay + 2x = 2$$
$$ay = -2x + 2$$
$$y = -\frac{2}{a}x + \frac{2}{a} \qquad m_{\perp} = \frac{a}{2}$$

$$y'\big|_4 = 2(4) - \frac{1}{\sqrt{4}} = 8 - \frac{1}{2} = \frac{15}{2}$$

$$\frac{a}{2} = \frac{15}{2}$$

$$a = 15$$

$f(x) = 2x^3\sqrt{x} + x^3 - 4x\sqrt{x} - 2x$

$= 2x^{7/2} + x^3 - 4x^{3/2} - 2x$

7. If $f(x) = (x^3 - 2x)(2\sqrt{x} + 1)$, find $f'(x)$ two ways: rewriting $f(x)$ and differentiating, and using the Product Rule.

$f'(x) = 7x^{5/2} + 3x^2 - 6x^{1/2} - 2$

product rule

$f'(x) = (3x^2 - 2)(2\sqrt{x} + 1) + (x^3 - 2x)(x^{-\frac{1}{2}})$

8. Find each value of x at which $f(x) = (1 - x)^5(5x + 2)^4$ has a horizontal tangent line.

$f'(x) = -5(1-x)^4(5x+2)^4 + 4(1-x)^5(5x+2)^3(5)$

$= -5(1-x)^4(5x+2)^3(5x+2 - 4(1-x))$

$= -5(1-x)^4(5x+2)^3(5x+2-4+4x)$

$0 = -5(1-x)^4(5x+2)(9x+2)$

$x = 1, -\frac{2}{5}, -\frac{2}{9}$

9. Let $f(x) = \dfrac{(\sqrt{x} - 1)^2}{x}$. Find $f'(x)$ and write as a single fraction. Write the equation of the tangent line to $f(x)$ at $x = 4$.

$f(x) = \dfrac{(\sqrt{x}-1)(\sqrt{x}-1)}{x} = \dfrac{x - 2\sqrt{x} + 1}{x} = 1 - 2x^{-\frac{1}{2}} + x^{-1}$

$f'(x) = x^{-3/2} - x^{-2}$

$= \dfrac{1}{x^{3/2}} - \dfrac{1}{x^2}$

$= \dfrac{x^{1/2}}{x^2} - \dfrac{1}{x^2} = \dfrac{\sqrt{x}-1}{x^2}$

$f'(4) = \dfrac{\sqrt{4}-1}{4^2} = \dfrac{1}{16}$

$f(4) = \dfrac{(\sqrt{4}-1)^2}{4} = \dfrac{1}{4}$

$\frac{1}{4} = \frac{1}{16}(4) + b \quad b = 0$

$\boxed{y = \frac{1}{16}x}$

$f(3) = 1$

10. Write the equation of the tangent line to $f(x) = \left(x - \frac{6}{x}\right)^3$ at $x = 3$.

$f'(x) = 3\left(x - \frac{6}{x}\right)^2\left(1 + \frac{6}{x^2}\right)$ $\quad$ $f'(3) = 3\left(3 - \frac{6}{3}\right)^2\left(1 + \frac{6}{9}\right)$

$= 3(1)\left(\frac{15}{9}\right) = 5$

$y = mx + b$

$1 = 5(3) + b \Rightarrow b = -14$

$y = 5x - 14$

11. Find each value of x at which the function $f(x) = \frac{\sqrt[3]{6x+1}}{x}$ has

(a) horizontal and (b) vertical tangent lines.

Write the equation of each of those lines.

$f(x) = \frac{(6x+1)^{1/3}}{x}$ $\quad$ $f'(x) = \frac{\frac{1}{3}(6x+1)^{-\frac{2}{3}} 6x - (6x+1)^{1/3}}{x^2}$

$= \frac{\frac{2x}{(6x+1)^{2/3}} - (6x+1)^{1/3}}{x^2}$

$= \frac{2x - (6x+1)}{(6x+1)^{2/3}x^2} = \frac{-4x-1}{(6x+1)^{2/3}x^2}$

horizontal tangent line at $x = -\frac{1}{4}$ $\quad$ $f\left(\frac{1}{4}\right) = 2^{5/3}$

$y = 2^{5/3}$

vertical tangent line $\quad x = -\frac{1}{6}, 0$

12. Suppose that $f(4) = -1$, $g(4) = 2$, $f(-4) = 1$, $g(-4) = 3$, $f'(4) = -2$, $g'(4) = 12$, $f'(-4) = 6$, and $g'(-1) = -2$.

Find: (a) $h'(4)$ if $h(x) = g(f(x))$ and (b) $H'(4)$ if $H(x) = \sqrt{xf(x) + \frac{x^2}{2}}$.

a) $h'(x) = g'(f(x))\, f'(x)$

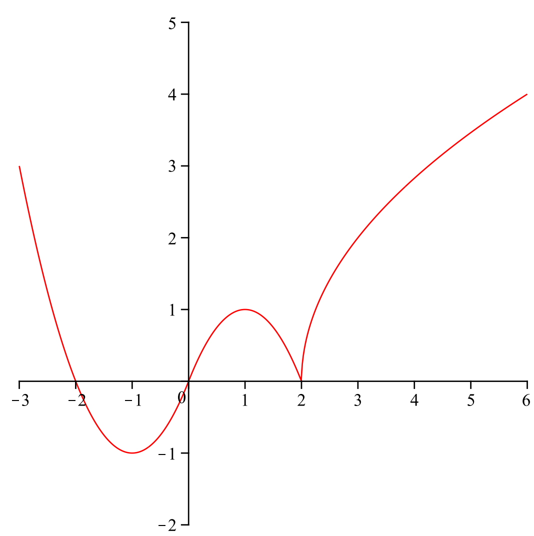
$h'(4) = g'(f(4))\, f'(4) = g'(-1)(-2) = -2(-2) = 4$

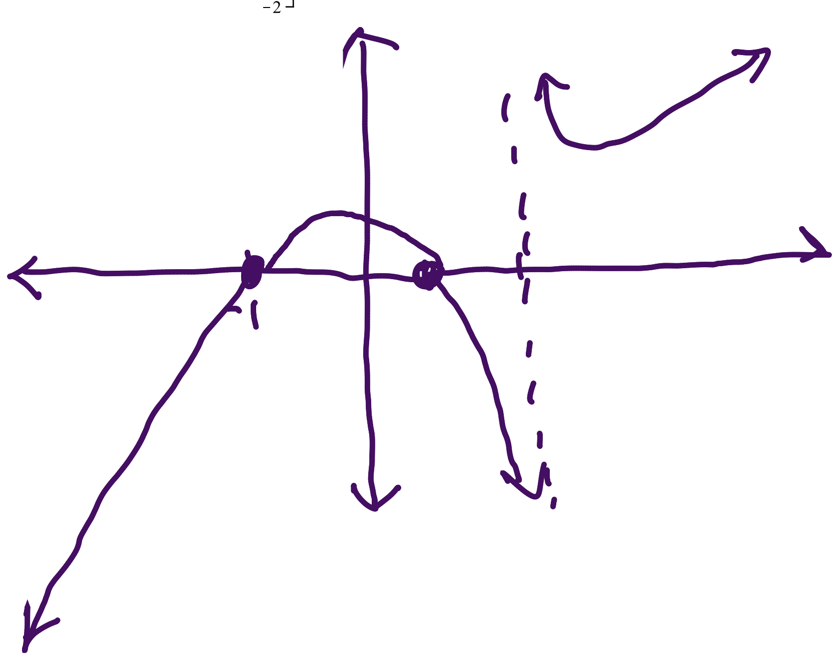
b) $H'(x) = \frac{1}{2}\left(x f(x) + \frac{x^2}{2}\right)^{-\frac{1}{2}} (f(x) + xf'(x) + x)$

$H'(4) = \frac{1}{2}\left(4 f(4) + \frac{16}{2}\right)^{-\frac{1}{2}} (f(4) + 4f'(4) + 4)$

$= \frac{1}{2} \frac{1}{(4(-1)+8)^{1/2}} (-1 + 4(-2) + 4) = \frac{1}{2} \cdot \frac{1}{2}(-5) = -\frac{5}{4}$

13. Sketch a possible graph of the derivative of the function $y = f(x)$ shown below.





14. Find the derivative:

(a) $f(x) = 3^{2x-1}$ (b) $f(x) = \log_4(x^2 - x)$

a)

$$f'(x) = 2\ln(3)\,3^{2x-1}$$

b) $$f'(x) = \frac{2x-1}{\ln(4)(x^2-x)}$$

15. Find the slope of the tangent line to the curve given by $\sqrt{3x-y} - e^{x+y} = 1 + \ln x$ at $(1, -1)$.

$$\frac{1}{2}(3x-y)^{-\frac{1}{2}}\left(3 - \frac{dy}{dx}\right) - e^{x+y}\left(1 + \frac{dy}{dx}\right) = \frac{1}{x}$$

$$\frac{1}{2}(3-(-1))^{-\frac{1}{2}}(3-m) - e^{0}(1+m) = 1$$

$$\frac{1}{4}(3-m) - (1+m) = 1$$

$$\frac{3}{4} - \frac{1}{4}m - 1 - m = 1$$

$$-\frac{5}{4} = \frac{5}{4}m$$

$$m = -1$$

16. Find the first three derivatives of $f(x) = \ln(x^2 + 1)$.

$$f'(x) = \frac{2x}{x^2+1}$$

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

$$f^{(3)}(x) = \frac{-4x(x^2+1)^{\cancel{2}\,1} - 2(2-2x^2)\cancel{(x^2+1)}(2x)}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-4x(x^2+1) - 4x(2-2x^2)}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3}$$

$f(x) = \ln(2x-6) - \frac{1}{2}\ln(x^2+3)$

17. Find each value of x at which $f(x) = \ln\left(\frac{2x-6}{\sqrt{x^2+3}}\right)$ has a horizontal tangent line.

$f'(x) = \frac{2}{2x-6} - \frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+3)}$

$0 = \frac{1}{x-3} - \frac{x}{x^2+3}$

$0 = x^2+3 - (x-3)x$

$0 = x^2+3 - x^2+3x$

$0 = 3+3x$

$x = -1$

-1 is not in the domain of f

no horizontal tangent lines

18. Find $f'(x)$ if $f(x) = \ln \frac{e^{x-3}\sqrt[3]{6+3x}}{(3x+1)^2}$.

$f(x) = \ln e^{x-3} + \frac{1}{3}\ln(6+3x) - 2\ln(3x+1)$

$= 1 + \frac{3}{3(6+3x)} - \frac{2(3)}{3x+1}$

$= 1 + \frac{1}{6+3x} - \frac{6}{3x+1}$

19. Use Logarithmic Differentiation to find the slope of the tangent line to $f(x) = x^{\sqrt{x}}$ at $x = 4$.

$y = x^{\sqrt{x}}$

$\ln y = \ln x^{\sqrt{x}}$

$= \sqrt{x}\ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\ln x + \frac{\sqrt{x}}{x}$

$\frac{dy}{dx} = y\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$

$= x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$

$f'(4) = 4^2\left(\frac{\ln(4)}{2\cdot 2} + \frac{1}{2}\right)$

$= 16\left(\frac{\ln 4}{4} + \frac{1}{2}\right)$

$= 4\ln 4 + 8$

- next exam

20. Find each value at which $f(x) = \dfrac{x^4}{4} - \dfrac{x^3}{3} - x^2$ has a relative maximum or minimum.

    Find the absolute extrema of $f(x)$ on $[-2, 1]$.

21. Let $f(x) = \dfrac{\sqrt[3]{3x-2}}{x}$.

    (a) Find $f'(x)$ and write as a single fraction.

    (b) Find the equation of each horizontal and vertical tangent line of $f(x)$.

    (c) Find each x-value at which $f(x)$ has a critical number.

    (d) ~~Find the relative extreme values of $f(x)$.~~

$$f'(x) = \frac{\frac{1}{\cancel{3}}(3x-2)^{-\frac{2}{3}}(\cancel{3})\,x - (3x-2)^{\frac{1}{3}}}{x^2}$$

$$= \frac{\frac{x}{(3x-2)^{2/3}} - (3x-2)^{1/3}}{x^2}$$

$$= \frac{x - (3x-2)}{x^2(3x-2)^{2/3}}$$

$$= \frac{-2x+2}{x^2(3x-2)^{2/3}}$$

horizontal tangent line at $x=1$ $f(1)=1$ $y=1$

vertical tangent line $x = 2/3$

critical numbers at $x = 1, 2/3$

22. The cost function for a product is
$C(x) = 1.25x^2 + 25x + 8000.$

(a) Suppose the company decides to increase production by 4 units per day when the current daily production level is 50 units. Find the rate of change of cost with respect to time.

(b) If $C(x) = 1.25x^2 + 25x + 8000$, find each interval on which **average cost** is increasing and decreasing. For what production level x is average cost minimized?

a)

$\frac{dx}{dt} = 4 \qquad \frac{dC}{dt} = ?$ when $x = 50$

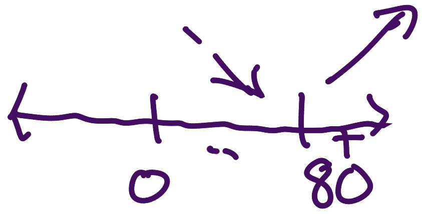
$$\frac{dC}{dt} = \frac{5}{2}x\frac{dx}{dt} + 25\frac{dx}{dt}$$

$$\frac{dC}{dt} = \frac{5}{2}(50)(4) + 25(4) = 500 + 100 = \$600 \text{ / day}$$

b) $$\bar{C}(x) = \frac{1.25x^2 + 25x + 8000}{x} = 1.25x + 25 + \frac{8000}{x}$$

$$\bar{C}'(x) = 1.25 - \frac{8000}{x^2} = 0$$

$$1.25x^2 = 8000$$
$$x^2 = 6400$$
$$x = 80$$



decreasing on $(0, 80)$

increasing for $x > 80$

$p(7500) = -.02(7500) + 400$

at \$250

23. The demand function for a certain product is given by

$p(x) = -0.02x + 400$, $0 \leq x \leq 20,000$, where p is the unit price when x items are sold. The cost function for the product is

$C(x) = 100x + 300,000$.

(a) Find the marginal profit of the product when $x = 2000$.

(b) Find the actual profit from the sale of the 2001st item. Compare to your answer in (a).

(c) Find each interval on which the profit function $P(x) = -0.02x^2 + 300x - 300,000$ is increasing and decreasing. Remember that $0 \leq x \leq 20,000$. How many items should be sold to maximize profit? At what price?

a) Profit = revenve − cost

$$P(x) = xp(x) - C(x)$$
$$= x(-.02x + 400) - (100x + 300,000)$$
$$= -.02x^2 + 400x - 100x - 300,0000$$
$$= -.02x^2 + 300x - 300,000$$
$$P'(x) = -.04x + 300$$
$$P'(2000) = -.04(2000) + 300 = 220$$

b) $\Delta P = P(2001) - P(2000) = 219.98$

c) $0 = -.04x + 300$ $\quad x = 7500$

increasing $(0, 7500)$ decreasing $(7500, 20,000)$

Profit is maximized when 7500 are sold