1) Find $\text{div}(\vec{F})$ & $\text{curl}(\vec{F})$. Determine if $\vec{F}$ is conservative and find its potential function if it is.

① $\vec{F} = \langle y, x+z, y \rangle$

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle y, x+z, y \rangle$$

$$= \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial y}[x+z] + \frac{\partial}{\partial z}[y] = 0$$

$$\text{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x+z & y \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\vec{F}$ is conservative because $\text{curl}\vec{F} = 0$.

$$\int y\,dx = yx + p(y,z)$$

$$\int x+z\,dy = xy + zy + q(x,z)$$

$$\int y\,dz = yz + r(x,y)$$

If we say $p(y,z) = yz$, $q(x,z) = 0$, $r(x,y) = xy$ then all the integrals match and give

$$f(x,y,z) = xy + yz$$

② $\vec{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$

$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = 4e^z - \sin(y) + 2x^2e^z$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix} = \langle 0,0,0 \rangle$$

$\vec{F}$ is conservative.

$\int 4xe^z\,dx = 2x^2e^z + p(y,z)$ | $p(y,z) = \sin(y)$

$\int \cos(y)\,dy = \sin(y) + q(x,z)$ | $q(x,z) = 2x^2e^z$

$\int 2x^2e^z\,dz = 2x^2e^z + r(x,y)$ | $r(x,y) = \sin(y)$

$$f(x,y,z) = 2x^2e^z + \sin(y)$$

③ $\vec{F} = \langle xe^{2x}, ye^{2z}, ze^{2y} \rangle$

$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = e^{2x} + 2xe^{2x} + e^{2z} + e^{2y}$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{2x} & ye^{2z} & ze^{2y} \end{vmatrix}$$

$$= \langle 2ze^{2y} - 2ye^{2z}, 0, 0 \rangle$$

$\vec{F}$ is <u>not</u> conservative

④ $\vec{F} = \langle \frac{y}{1+x^2}, \arctan(x), 2z \rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left[\frac{y}{1+x^2}\right] + \frac{\partial}{\partial y}\left[\arctan x\right] + \frac{\partial}{\partial z}\left[2z\right]$$

$$= \frac{-2xy}{(1+x^2)^2} + 2$$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{y}{1+x^2} & \arctan(x) & 2z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\vec{F}$ is conservative.

$\int \frac{y}{1+x^2}\,dx = y\arctan(x) + p(y,z)$ | $p(y,z) = z^2$

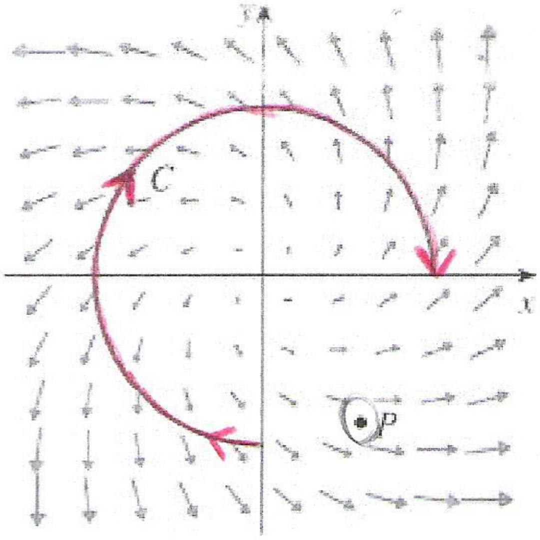
$\int \arctan(x)\,dy = y\arctan(x) + q(x,z)$ | $q(x,z) = z^2$

$\int 2z\,dz = z^2 + r(x,y)$ | $r(x,y) = y\arctan(x)$

$$f(x,y,z) = y\arctan(x) + z^2$$

2)

① Is $\int_C \vec{F}\cdot d\vec{r}$ positive, negative, or zero?



Negative, the vector field generally points opposite the path C.

② Is div$\vec{F}$ at P positive, negative, or zero?

Positive. Some small vectors point towards P, but more big vectors point away. "Divergence" literally means "going away" so more flow away means positive divergence.

3) ① $\int_C y\,dx + (x+y^2)\,dy$ if C is the ellipse $4x^2+9y^2=36$ with counterclockwise orientation.

C is closed, so use Green's theorem.

$$\oint_C y\,dx+(x+y^2)\,dy = \iint_R \frac{\partial}{\partial x}[x+y^2] - \frac{\partial}{\partial y}[y]\,dA$$

$$= \iint_R 1-1\,dA = 0$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA \;;\; \vec{F}=\langle P,Q\rangle$$

② $\int_C (x^2+y^2+z^2)\,ds$ where C is $\vec{r}(t)=\langle t, \cos(2t), \sin(2t)\rangle$ and $0 \le t \le 2\pi$

$ds = |\vec{r}\,'|\,dt$

$\vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$

$|\vec{r}\,'| = \sqrt{1^2 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{5}$

$\int_0^{2\pi} (t^2 + \cos^2(2t) + \sin^2(2t))\sqrt{5}\,dt = \sqrt{5}\int_0^{2\pi} t^2+1\,dt = \sqrt{5}\ \frac{t^3}{3}+t\Big|_0^{2\pi}$

$= \sqrt{5}\left(\frac{8\pi^3}{3} + 2\pi\right)$

③ $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle (1+xy)e^{xy}, x^2e^{xy}\rangle$ where C is a curve moving from $(1,0)$ to $(0,2)$.

Because we don't know the shape of C, $\vec{F}$ is probably conservative. We could check and then find the potential function f, or just find f right away.

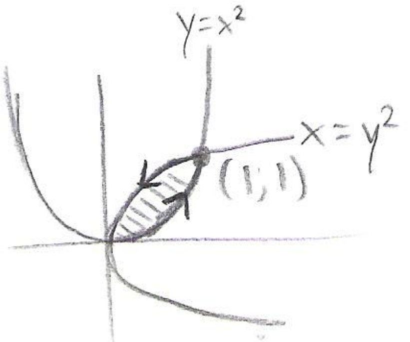
$\int (1+xy)e^{xy}\,dx = xe^{xy} + g(y)$ | $g(y)=0$

$\int x^2e^{xy}\,dy = xe^{xy} + h(x)$ | $h(x)=0$

$f(x,y) = xe^{xy}$

$$\int_C \vec{F}\cdot d\vec{r} = xe^{xy}\Big|_{(1,0)}^{(0,2)} = 0\cdot e^{0\cdot 2} - 1\cdot e^{1\cdot 0} = -1$$

④ $\int_C (y+e^{\sqrt{x}})dx + (2x+\cos(y^2))dy$ where C is the positively oriented boundary curve of the region between $y=x^2$ and $x=y^2$.



The curve is closed, so we try Green's Theorem.

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

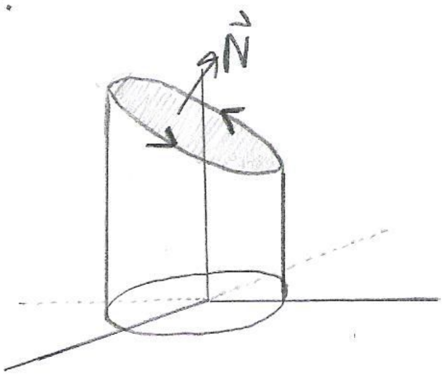
$$= \iint \frac{\partial}{\partial x}\left[2x+\cos(y^2)\right] - \frac{\partial}{\partial y}\left[y+e^{\sqrt{x}}\right] dA = \iint_R 1\cdot dA$$

$$= \int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx = \int_0^1 \sqrt{x}-x^2\,dx = \frac{2x^{3/2}}{3} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

⑤ $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=\langle x+yz,\ 2yz,\ x-y\rangle$ and C is the intersection of $x^2+y^2=4$ and $x+y+z=1$ with counterclockwise orientation when viewed from above.

Since C is closed, use Stokes' Theorem.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\vec{N}\,dS$$



where C is CCW and $\vec{N}$ is pointing upward.

S is any surface having C as a boundary.

$$\text{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+yz & 2yz & x-y \end{vmatrix} = \langle -1-2y,\ y-1,\ -z\rangle$$

Our surface S will be the portion of the plane $\underset{(\text{alt. } z=1-x-y)}{x+y+z=1}$ restricted to the region $x^2+y^2=4$.

Since our surface is a function $z=g(x,y)$, we can compute

$$\iint_S \text{curl}\vec{F}\cdot\vec{N}\,dS = \iint_R \text{curl}\vec{F}\cdot\langle -g_x, -g_y, 1\rangle\,dA$$

where R is the xy-region under S.

Clearly our R is the circle of radius 2.

$$\iint_R \text{curl}\vec{F}\cdot\langle -g_x, -g_y, 1\rangle\, dA = \iint_R \langle -2y-1, y-1, -z\rangle \cdot \langle 1, 1, 1\rangle\, dA$$

$$= \iint_R -y-z-2\, dA = \iint_R -y-(1-y-x)-2\, dA = \iint_R x-3\, dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\cos\theta - 3r\, dr\, d\theta = \int_0^{2\pi} \frac{r^3}{3}\cos\theta - \frac{3r^2}{2}\Big|_0^2\, d\theta$$

$$= \int_0^{2\pi} \frac{8}{3}\cos\theta - 6\, d\theta = \frac{8}{3}\sin\theta - 6\theta\Big|_0^{2\pi} = -12\pi$$

4) ① $\iint_S (x^2z+y^2z)\,dS$, where S is the part of the plane $z=4+x+y$ inside the cylinder $x^2+y^2=4$.

Since $z=g(x,y)$, $dS = \sqrt{g_x^2+g_y^2+1}\, dA$.

$$\iint_S (x^2+y^2)z\, dS = \iint_R (x^2+y^2)(4+x+y)\sqrt{1+1+1}\, dA$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 r^2(r\cos\theta + r\sin\theta + 4)\, r\,dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi} \frac{r^5}{5}\cos\theta + \frac{r^5}{5}\sin\theta + r^4\Big|_0^2\, d\theta = \sqrt{3}\int_0^{2\pi} \frac{32}{5}(\sin\theta + \cos\theta) + 16\, d\theta$$

$$= \sqrt{3}\left[\frac{32}{5}(-\cos\theta + \sin\theta) + 16\theta\right]_0^{2\pi} = 32\pi\sqrt{3}$$

② $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=\langle xz, -2y, 3x\rangle$ and S is the sphere $x^2+y^2+z^2=4$ with outward orientation.

The divergence theorem lets us compute flux integrals over closed surfaces.

$$\oiint_S \vec{F}\cdot\vec{N}\,dS = \iiint_Q \operatorname{div}\vec{F}\,dV$$ where Q is the region inside of S.

$\operatorname{div}\vec{F} = z-2$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi}\int_0^{2}(\rho\cos\phi-2)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_0^{\pi}\int_0^{2} \rho^3\sin\phi\cos\phi-2\rho^2\sin\phi\,d\rho\,d\phi$$

$$= 2\pi\int_0^{\pi} \frac{\rho^4}{4}\sin\phi\cos\phi-\frac{2\rho^3}{3}\sin\phi\Big|_0^2\,d\phi$$

$$= 2\pi\int_0^{\pi} 4\sin\phi\cos\phi-\frac{16}{3}\sin\phi\,d\phi = 2\pi\int_0^{\pi} 2\sin(2\phi)-\frac{16}{3}\sin\phi\,d\phi$$

$$= 2\pi\left(-\cos(2\phi)+\frac{16}{3}\cos(\phi)\right|_0^{\pi} = 2\pi\left(\cos(2\phi)-\frac{16}{3}\cos(\phi)\right|_{\pi}^{0}$$

$$= 2\pi\left[\left(1-\frac{16}{3}\right)-\left(1+\frac{16}{3}\right)\right] = \frac{-64\pi}{3}$$

③ $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle x^2, xy, z\rangle$ and S is the part of the paraboloid $z = x^2+y^2$ below the plane $z=1$ with upward orientation.

$\iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$ where $\vec{r}(u,v)$ parameterizes S. For us, $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$.

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$$

On this $\vec{r}$, $\vec{F} = \langle r^2\cos^2\theta, r^2\sin\theta\cos\theta, r^2\rangle$

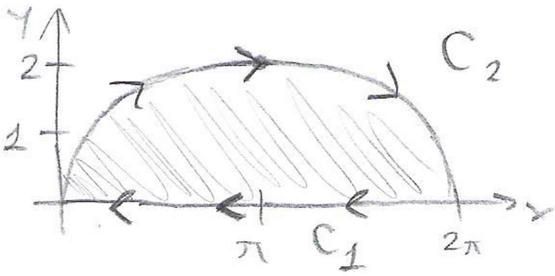
So $\vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta) = -2r^4\cos^3\theta - 2r^4\sin^2\theta\cos\theta + r^3$

$$= -2r^4\cos\theta(\cos^2\theta + \sin^2\theta) + r^3$$
$$= r^3 - 2r^4\cos\theta$$

$$\int_0^{2\pi}\int_0^1 r^3 - 2r^4\cos\theta\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4} - \frac{2r^5}{5}\cos\theta\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} - \frac{2}{5}\cos\theta\,d\theta = \frac{\theta}{4} - \frac{2}{5}\sin\theta\Big|_0^{2\pi} = \frac{\pi}{2}$$

5) Find the area of the region between the x-axis and the cycloid $x=t-\sin t$, $y=1-\cos t$, $0\le t\le 2\pi$.



Using Green's Theorem, $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

To find the area, we invent an "artificial" vector field so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, so we get $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_R dA = \text{Area}$

Let's use $\vec{F} = \langle 0, x\rangle$, and $C = C_1 \cup C_2$ as in the picture.

$\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int 0\,dx + x\,dy = \int x\,dy = 0$ because on the path C_1, y isn't changing.

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int 0\,dx + x\,dy = \int_0^{2\pi} (t-\sin t)(\sin t)\,dt = \int_0^{2\pi} t\sin t - \sin^2 t\,dt$$

$$= \underbrace{\int_0^{2\pi} t\sin t\,dt}_{\text{I.B.P}} - \underbrace{\int_0^{2\pi} \sin^2 t\,dt}_{\sin^2 t = \frac{1-\cos(2t)}{2}} = -2\pi - \pi = -3\pi$$

Green's Theorem requires C to be counterclockwise. Our C is clockwise, so Area $= 3\pi$

6) Consider the parametric surface S: $\vec{r}(u,v) = \langle v^2, -uv, u^2 \rangle$,
$0 \leq u \leq 3$, $-3 \leq v \leq 3$.

① Find an equation of the tangent plane to the surface at the point $(4,-2,1)$.

We should figure out which (u,v) corresponds to the point $(4,-2,1)$.

$$\begin{array}{lll} v^2 = 4 & & v = \pm 2 \\ -uv = -2 & \Rightarrow & uv = 2 \\ u^2 = 1 & & u = \pm 1 \end{array}$$

But we are given $0 \leq u \leq 3$, so then $u=1$ and $v=2$.

Now to make a plane, we need a normal vector. We'll use $\vec{r}_u \times \vec{r}_v$ at $(u,v) = (1,2)$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = \langle 2u^2, 4uv, 2v^2 \rangle$$

$(\vec{r}_u \times \vec{r}_v)(1,2) = \langle 2, 8, 8 \rangle \Rightarrow \langle 1, 4, 4 \rangle$ just for simplicity

The plane will look like $x + 4y + 4z + d = 0$, and we know it passes through $(4,-2,1)$

$4 + 4(-2) + 4(1) + d = 0 \Rightarrow 4 - 8 + 4 + d = 0 \Rightarrow d = 0$

$$x + 4y + 4z = 0$$

② Set up an integral for the surface area of S.

$$\iint_S dS = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$$

From part 1, we know $\vec{r}_u \times \vec{r}_v = \langle 2u^2, 4uv, 2v^2 \rangle$.

So $|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4 + 16u^2v^2 + 4v^2} = 2\sqrt{u^4 + 4u^2v^2 + v^4}$

$$S.A. = \int_0^3 \int_{-3}^3 2\sqrt{u^4 + 4u^2v^2 + v^4} \, dv\,du$$

7) Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ such that $\operatorname{curl}\vec{G} = \langle x, y, z \rangle$?

It is a fact (which you can easily check) that for any vector field $\vec{F}$ with differentiable components, $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$.

$\operatorname{div}(\operatorname{curl}(\vec{G})) = \operatorname{div}(\langle x, y, z \rangle) = 1 + 1 + 1 \neq 0$ so no such $\vec{G}$ exists.

8) Find the work done by the force field $\vec{F} = \langle z, x, y \rangle$ in moving a particle from the point $(3,0,0)$ to the point $(0, \pi/2, 3)$ along

① A straight line.

We need to parameterize the path.

$$\vec{r}(t) = \langle 3-3t, \tfrac{\pi}{2}t, 3t \rangle, \quad 0 \le t \le 1$$

$$\text{work} = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{r}\,' dt = \int_0^1 \langle 3t, 3-3t, \tfrac{\pi}{2}t \rangle \cdot \langle -3, \tfrac{\pi}{2}, 3 \rangle dt$$

$$= \int_0^1 -9t + \frac{3\pi}{2} - \frac{3\pi}{2}t + \frac{3\pi}{2}t \, dt = \int_0^1 \frac{3\pi}{2} - 9t \, dt = \frac{3\pi}{2}t - \frac{9t^2}{2} \Big|_0^1$$

$$= \frac{3\pi}{2} - \frac{9}{2} = \frac{1}{2}(3\pi - 9)$$

② The helix $x = 3\cos t$, $y = t$, $z = 3\sin t$.

$$\vec{r}(t) = \langle 3\cos t, t, 3\sin t \rangle, \quad 0 \le t \le \pi/2$$

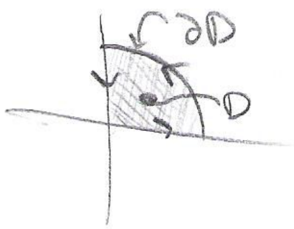
$$\text{work} = \int_0^{\pi/2} \langle 3\sin t, 3\cos t, t \rangle \cdot \langle -3\sin t, 1, 3\cos t \rangle dt$$

$$= \int_0^{\pi/2} \underbrace{-9\sin^2 t}_{\sin^2 t = \frac{1-\cos 2t}{2}} + 3\cos t + \underbrace{3t\cos t}_{\text{I.B.P}} \, dt = -\frac{3\pi}{4}$$

9) Let $\vec{F} = \langle x^2 - y^2, 2xy \rangle$ be the velocity field of a two-dimensional fluid flow. If D is the region in the first quadrant bounded by $y = \sqrt{1-x^2}$, $x=0$, $y=0$ with its boundary ∂D oriented counterclockwise, find:

① the circulation of $\vec{F}$ around ∂D.

This is $\int_{\partial D} \vec{F} \cdot d\vec{r}$. Since ∂D is closed, we use Green's theorem.



$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}[2xy] - \frac{\partial}{\partial y}[x^2 - y^2] \, dA = \iint_D 2y + 2y \, dA$$

$$= \int_0^{\pi/2}\int_0^1 4r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi/2} \frac{4r^3}{3}\Big|_0^1 \sin\theta \, d\theta = \int_0^{\pi/2} \frac{4}{3}\sin\theta \, d\theta$$

$$= -\frac{4}{3}(\cos\theta|_0^{\pi/2} = -\frac{4}{3}(0-1) = 4/3$$

② the flux of $\vec{F}$ through ∂D

This is $\int \vec{F} \cdot \vec{N} \, ds$. Green's theorem can be used to show that this is equal to $\iint \operatorname{div}\vec{F} \, dA$.

$$\int_{\partial D} \vec{F} \cdot \vec{N} \, ds = \iint_D \frac{\partial}{\partial x}[x^2 - y^2] + \frac{\partial}{\partial y}[2xy] = \iint_D 4x \, dA$$

$$= \int_0^{\pi/2}\int_0^1 4r^2 \cos\theta \, dr \, d\theta = \int_0^{\pi/2} \frac{4}{3}\cos\theta \, d\theta = \frac{4}{3}(\sin\theta|_0^{\pi/2}$$

$$= 4/3$$

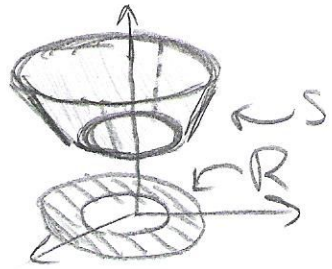
10) Compute the flux of $\vec{F}$ across the given surface.

① $\vec{F} = \langle \sin y, \sin z, yz \rangle$; S is the rectangular surface $0 \le y \le 2$, $0 \le z \le 3$ in the yz-plane with normal vector pointing in the negative x-direction.

$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iint_R \langle \sin y, \sin z, yz \rangle \cdot \langle -1, 0, 0 \rangle \, dA$$

$$= \int_0^2 \int_0^3 -\sin y \, dz\, dy = 3\int_0^2 -\sin y\, dy = 3\left(\cos y\right|_0^2 = 3(\cos(2) - 1)$$

② $\vec{F} = \langle -x, -y, z^3 \rangle$; S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z=1$ & $z=3$ with downward orientation.



$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

We parameterize the surface by

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle, \quad 0 \le \theta \le 2\pi, \quad 1 \le r \le 3$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \langle r\cos\theta, r\sin\theta, -r \rangle$$

this does point downward so no need to change it

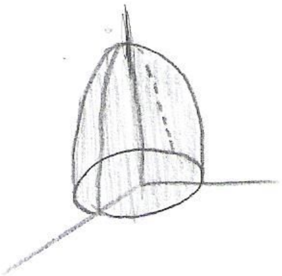
$$\iint_R \langle -r\cos\theta, -r\sin\theta, r^3\rangle \cdot \langle r\cos\theta, r\sin\theta, -r\rangle \, dA$$

$$= \iint -r^2\cos^2\theta - r^2\sin^2\theta - r^4 \, dA = \iint -r^2 - r^4 \, dA$$

$$= -\int_0^{2\pi}\int_1^3 r^4 + r^2 \, dr\, d\theta = -2\pi \left(\frac{r^5}{5} + \frac{r^3}{3}\right|_1^3 = -2\pi\left[\left(\frac{243}{5}+\frac{27}{3}\right) - \left(\frac{1}{5}+\frac{1}{3}\right)\right]$$

$$= -2\pi\left[\frac{729+135-3-5}{15}\right] = \frac{-1712}{15}\pi$$

③ $\vec{F} = \langle 2x^3+y^3, y^3+z^3, 3y^2z\rangle$; S is the surface of the solid bounded by the paraboloid $z = 1-x^2-y^2$ and the xy-plane.



Since S is closed, we can use Divergence Theorem.

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint 6x^2+3y^2+3y^2\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^3\,dz\,dr\,d\theta = 2\pi\int_0^1 6r^3(1-r^2)\,dr = 2\pi\int_0^1 6r^3 - 6r^5\,dr$$

$$= 2\pi\left(\frac{3r^4}{2} - r^6\right|_0^1 = 2\pi\left(\frac{3}{2}-1\right) = \pi$$

11) True or false

① If $\vec{F} = \langle P, Q \rangle$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in an open region D, then $\vec{F}$ is conservative.

False. D has to be simply connected.

② $\int_{-C} f(x,y)\,ds = -\int_C f(x,y)\,ds$

False. A scalar line integral does not change when the path is reversed.

③ $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

True.

④ If S is a sphere and $\vec{F}$ is a constant vector field, $\iint_S \vec{F}\cdot d\vec{S} = 0$.

True. S is closed, so use Div. Thm. $\operatorname{div}\vec{F} = 0$.

⑤ The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve C is $\oint_C y\,dx$.

False. By Green's Theorem, $\oint_C y\,dx = \iint_R \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}[y]\,dA = \iint_R -1\,dA$

$= -\text{area}$

⑥ The flux of curl $\vec{F}$ through every oriented surface is zero.

False. $\iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS = 0$ is just not always true.

It is true if $\vec{F}$ is conservative or if S is closed.