

1) Find  $\text{div}(\vec{F})$  &  $\text{curl}(\vec{F})$ . Determine if  $\vec{F}$  is conservative and find its potential function if it is.

$$\textcircled{1} \vec{F} = \langle y, x+z, y \rangle$$

$$\begin{aligned} \text{div} \vec{F} &= \vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle y, x+z, y \rangle \\ &= \frac{\partial}{\partial x} [y] + \frac{\partial}{\partial y} [x+z] + \frac{\partial}{\partial z} [y] = 0 \end{aligned}$$

$$\text{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x+z & y \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\vec{F}$  is conservative because  $\text{curl} \vec{F} = 0$ .

$$\int y dx = yx + p(y, z)$$

$$\int x+z dy = xy + zy + q(x, z)$$

$$\int y dz = yz + r(x, y)$$

If we say  $p(y, z) = yz$ ,  $q(x, z) = 0$ ,  $r(x, y) = xy$  then all the integrals match and give

$$f(x, y, z) = xy + yz$$

$$\textcircled{2} \vec{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$$

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = 4e^z - \sin(y) + 2x^2e^z$$

$$\text{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\vec{F}$  is conservative.

$$\int 4xe^z dx = 2x^2e^z + p(y, z) \quad | \quad p(y, z) = \sin(y)$$

$$\int \cos(y) dy = \sin(y) + q(x, z) \quad | \quad q(x, z) = 2x^2e^z$$

$$\int 2x^2e^z dz = 2x^2e^z + r(x, y) \quad | \quad r(x, y) = \sin(y)$$

$$f(x, y, z) = 2x^2e^z + \sin(y)$$

$$\textcircled{3} \vec{F} = \langle xe^{2x}, ye^{2z}, ze^{2y} \rangle$$

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = e^{2x} + 2xe^{2x} + e^{2z} + e^{2y}$$

$$\text{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{2x} & ye^{2z} & ze^{2y} \end{vmatrix}$$

$$= \langle 2ze^{2y} - 2ye^{2z}, 0, 0 \rangle$$

Pg 2 |  $\vec{F}$  is not conservative

$$\textcircled{4} \vec{F} = \left\langle \frac{y}{1+x^2}, \arctan(x), 2z \right\rangle$$

$$\begin{aligned} \operatorname{div} \vec{F} &= \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x} \left[ \frac{y}{1+x^2} \right] + \frac{\partial}{\partial y} [\arctan(x)] + \frac{\partial}{\partial z} [2z] \\ &= \frac{-2xy}{(1+x^2)^2} + 2 \end{aligned}$$

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{1+x^2} & \arctan(x) & 2z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\vec{F}$  is conservative.

$$\int \frac{y}{1+x^2} dx = y \arctan(x) + p(y, z)$$

$$p(y, z) = z^2$$

$$\int \arctan(x) dy = y \arctan(x) + q(x, z)$$

$$q(x, z) = z^2$$

$$\int 2z dz = z^2 + r(x, y)$$

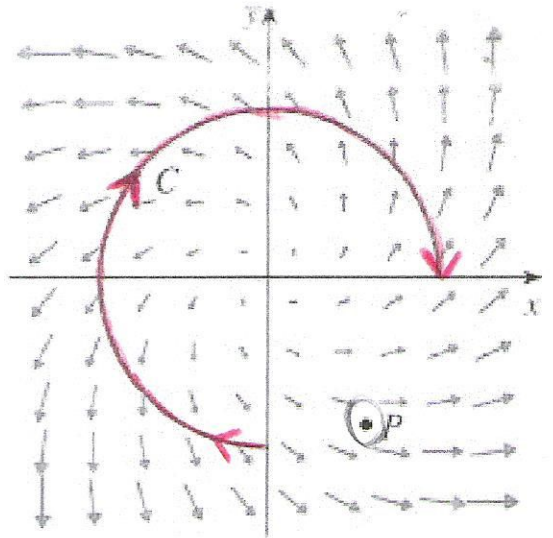
$$r(x, y) = y \arctan(x)$$

$$f(x, y, z) = y \arctan(x) + z^2$$

2)

① Is  $\int_C \vec{F} \cdot d\vec{r}$  positive, negative, or zero?

Negative, the vector field generally points opposite the path  $C$ .



② Is  $\text{div} \vec{F}$  at  $P$  positive, negative, or zero?

Positive. Some small vectors point towards  $P$ , but more big vectors point away. "Divergence" literally means "going away" so more flow away means positive divergence.

3) ①  $\int_C y dx + (x+y^2) dy$  if  $C$  is the ellipse  $4x^2 + 9y^2 = 36$  with counterclockwise orientation.

$C$  is closed, so use Green's theorem.

$$\oint_C y dx + (x+y^2) dy = \iint_R \frac{\partial}{\partial x} [x+y^2] - \frac{\partial}{\partial y} [y] dA$$

$$= \iint_R 1 - 1 dA = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA ; \vec{F} = \langle P, Q \rangle$$



$$\textcircled{2} \int_C (x^2 + y^2 + z^2) ds \text{ where } C \text{ is } \vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle \text{ and } 0 \leq t \leq 2\pi$$

$$ds = |\vec{r}'| dt$$

$$\vec{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$$

$$|\vec{r}'| = \sqrt{1^2 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{5}$$

$$\int_0^{2\pi} (t^2 + \cos^2(2t) + \sin^2(2t)) \sqrt{5} dt = \sqrt{5} \int_0^{2\pi} t^2 + 1 dt = \sqrt{5} \left. \frac{t^3}{3} + t \right|_0^{2\pi}$$

$$= \sqrt{5} \left( \frac{8\pi^3}{3} + 2\pi \right)$$

$$\textcircled{3} \int_C \vec{F} \cdot d\vec{r} \text{ where } \vec{F} = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle \text{ where}$$

$C$  is a curve moving from  $(1,0)$  to  $(0,2)$ .

Because we don't know the shape of  $C$ ,  $\vec{F}$  is probably conservative. We could check and then find the potential function  $f$ , or just find  $f$  right away.

$$\int (1+xy)e^{xy} dx = xe^{xy} + g(y)$$

$$g(y) = 0$$

$$\int x^2e^{xy} dy = xe^{xy} + h(x)$$

$$h(x) = 0$$

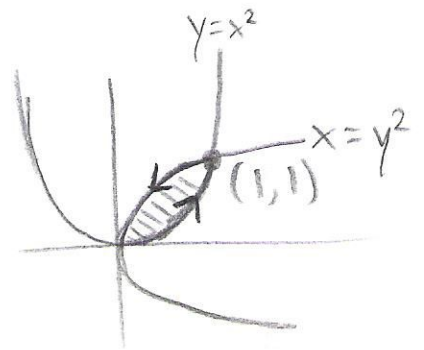
$$f(x,y) = xe^{xy}$$



$$\int_C \vec{F} \cdot d\vec{r} = x e^{xy} \Big|_{(1,0)}^{(0,2)} = 0 \cdot e^{0 \cdot 2} - 1 \cdot e^{1 \cdot 0} = -1$$

④  $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos(y^2)) dy$  where  $C$  is the positively oriented boundary curve of the region between  $y = x^2$  and  $x = y^2$ .

The curve is closed, so we try Green's Theorem.



$$\int_C P dx + Q dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

$$= \iint_R \frac{\partial}{\partial x} [2x + \cos(y^2)] - \frac{\partial}{\partial y} [y + e^{\sqrt{x}}] dA = \iint_R 1 dA$$

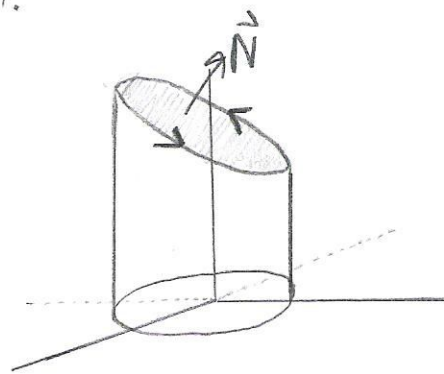
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} dy dx = \int_0^1 \sqrt{x} - x^2 dx = \frac{2x^{3/2}}{3} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

⑤  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = \langle x+yz, 2yz, x-y \rangle$  and  $C$  is the intersection of  $x^2+y^2=4$  and  $x+y+z=1$  with counterclockwise orientation when viewed from above.

Since  $C$  is closed, use Stokes' Theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \vec{N} dS$$

where  $C$  is ccw and  $\vec{N}$  is pointing upward.



$S$  is any surface having  $C$  as a boundary.

$$\text{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+yz & 2yz & x-y \end{vmatrix} = \langle -1-2y, y-1, -z \rangle$$

Our surface  $S$  will be the portion of the plane  $x+y+z=1$  restricted to the region  $x^2+y^2=4$ .  
(alt.  $z=1-x-y$ )

Since our surface is a function  $z=g(x,y)$ , we can compute

$$\iint_S \text{curl} \vec{F} \cdot \vec{N} dS = \iint_R \text{curl} \vec{F} \cdot \langle -g_x, -g_y, 1 \rangle dA$$

where  $R$  is the  $xy$ -region under  $S$ .



Clearly our  $R$  is the circle of radius 2.

$$\iint_R \text{curl } \vec{F} \cdot \langle -g_x, -g_y, 1 \rangle dA = \iint_R \langle -2y-1, y-1, -z \rangle \cdot \langle 1, 1, 1 \rangle dA$$

$$= \iint_R -y-z-2 dA = \iint_R -y-(1-y-x)-2 dA = \iint_R x-3 dA$$

$$= \int_0^{2\pi} \int_0^2 r^2 \cos \theta - 3r dr d\theta = \int_0^{2\pi} \left. \frac{r^3}{3} \cos \theta - \frac{3r^2}{2} \right|_0^2 d\theta$$

$$= \int_0^{2\pi} \left. \frac{8}{3} \cos \theta - 6 \right|_0^{2\pi} d\theta = \left. \frac{8}{3} \sin \theta - 6\theta \right|_0^{2\pi} = -12\pi$$

4) ①  $\iint_S (x^2z + y^2z) dS$ , where  $S$  is the part of the plane  $z = 4 + x + y$  inside the cylinder  $x^2 + y^2 = 4$ .

Since  $z = g(x, y)$ ,  $dS = \sqrt{g_x^2 + g_y^2 + 1} dA$ .

$$\iint_S (x^2 + y^2)z dS = \iint_R (x^2 + y^2)(4 + x + y) \sqrt{1 + 1 + 1} dA$$

$$= \sqrt{3} \int_0^{2\pi} \int_0^2 r^2 (r \cos \theta + r \sin \theta + 4) r dr d\theta$$

$$= \sqrt{3} \int_0^{2\pi} \left. \frac{r^5}{5} \cos \theta + \frac{r^5}{5} \sin \theta + r^4 \right|_0^2 d\theta = \sqrt{3} \int_0^{2\pi} \frac{32}{5} (\sin \theta + \cos \theta) + 16 d\theta$$

$$= \sqrt{3} \left[ \frac{32}{5} (-\cos \theta + \sin \theta) + 16\theta \right]_0^{2\pi} = 32\pi\sqrt{3}$$



②  $\iint_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = \langle xz, -2y, 3x \rangle$  and  $S$  is the sphere  $x^2 + y^2 + z^2 = 4$  with outward orientation.

The divergence theorem lets us compute flux integrals over closed surfaces.

$$\oiint_S \vec{F} \cdot \vec{N} dS = \iiint_Q \operatorname{div} \vec{F} dV \quad \text{where } Q \text{ is the region inside of } S.$$

$$\operatorname{div} \vec{F} = z - 2$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \int_0^2 (p \cos \phi - 2) p^2 \sin \phi dp d\phi d\theta$$

$$= 2\pi \int_0^\pi \int_0^2 p^3 \sin \phi \cos \phi - 2p^2 \sin \phi dp d\phi$$

$$= 2\pi \int_0^\pi \left. \frac{p^4}{4} \sin \phi \cos \phi - \frac{2p^3}{3} \sin \phi \right|_0^2 d\phi$$

$$= 2\pi \int_0^\pi 4 \sin \phi \cos \phi - \frac{16}{3} \sin \phi d\phi = 2\pi \int_0^\pi 2 \sin(2\phi) - \frac{16}{3} \sin \phi d\phi$$

$$= 2\pi \left( -\cos(2\phi) + \frac{16}{3} \cos(\phi) \right) \Big|_0^\pi = 2\pi \left( \cos(2\phi) - \frac{16}{3} \cos(\phi) \right) \Big|_\pi^0$$

$$= 2\pi \left[ \left( 1 - \frac{16}{3} \right) - \left( 1 + \frac{16}{3} \right) \right] = \frac{-64\pi}{3}$$

③  $\iint_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = \langle x^2, xy, z \rangle$  and  $S$  is the part of the paraboloid  $z = x^2 + y^2$  below the plane  $z = 1$  with upward orientation.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA \quad \text{where } \vec{r}(u, v)$$

parameterizes  $S$ . For us,  $\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle$ .  
 $0 \leq \theta \leq 2\pi$   
 $0 \leq r \leq 1$

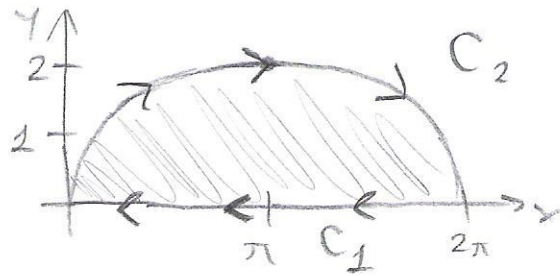
$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, r \rangle$$

On  $r$  this  $\vec{r}$ ,  $\vec{F} = \langle r^2 \cos^2 \theta, r^2 \sin \theta \cos \theta, r^2 \rangle$

$$\begin{aligned} \text{So } \vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta) &= -2r^4 \cos^3 \theta - 2r^4 \sin^2 \theta \cos \theta + r^3 \\ &= -2r^4 \cos \theta (\cos^2 \theta + \sin^2 \theta) + r^3 \\ &= r^3 - 2r^4 \cos \theta \end{aligned}$$

$$\begin{aligned} \iint_0^{2\pi} \int_0^1 r^3 - 2r^4 \cos \theta dr d\theta &= \int_0^{2\pi} \left. \frac{r^4}{4} - \frac{2r^5}{5} \cos \theta \right|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{4} - \frac{2}{5} \cos \theta d\theta = \left. \frac{\theta}{4} - \frac{2}{5} \sin \theta \right|_0^{2\pi} = \frac{\pi}{2} \end{aligned}$$

5) Find the area of the region between the x-axis and the cycloid  $x = t - \sin t$ ,  $y = 1 - \cos t$ ,  $0 \leq t \leq 2\pi$ .



Using Green's Theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

To find the area, we invent an "artificial" vector field so that  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ , so we get  $\oint_C \vec{F} \cdot d\vec{r} = \iint_R dA = \text{Area}$

Let's use  $\vec{F} = \langle 0, x \rangle$ , and  $C = C_1 \cup C_2$  as in the picture.

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int 0 dx + x dy = \int x dy = 0$  because on the path  $C_1$ ,  $y$  isn't changing.

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int 0 dx + x dy = \int_0^{2\pi} (t - \sin t)(\sin t) dt = \int_0^{2\pi} t \sin t - \sin^2 t dt$$

$$= \int_0^{2\pi} t \sin t dt - \int_0^{2\pi} \sin^2 t dt = -2\pi - \pi = -3\pi$$

$\underbrace{\int_0^{2\pi} t \sin t dt}_{\text{I.B.P}}$ 
 $\underbrace{\int_0^{2\pi} \sin^2 t dt}_{\sin^2 t = \frac{1 - \cos(2t)}{2}}$

Green's Theorem requires  $C$  to be counterclockwise.

Our  $C$  is clockwise, so  $\text{Area} = 3\pi$

6) Consider the parametric surface  $S: \vec{r}(u,v) = \langle v^2, -uv, u^2 \rangle$ ,  
 $0 \leq u \leq 3, -3 \leq v \leq 3$ .

① Find an equation of the tangent plane to the surface at the point  $(4, -2, 1)$ .

We should figure out which  $(u,v)$  corresponds to the point  $(4, -2, 1)$ .

$$\begin{array}{l} v^2 = 4 \\ -uv = -2 \\ u^2 = 1 \end{array} \Rightarrow \begin{array}{l} v = \pm 2 \\ uv = 2 \\ u = \pm 1 \end{array} \quad \begin{array}{l} \text{But we are given } 0 \leq u \leq 3, \\ \text{so then } u=1 \text{ and } v=2. \end{array}$$

Now to make a plane, we need a normal vector. We'll use  $\vec{r}_u \times \vec{r}_v$  at  $(u,v) = (1,2)$ .

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = \langle 2u^2, 4uv, 2v^2 \rangle$$

$$(\vec{r}_u \times \vec{r}_v)(1,2) = \langle 2, 8, 8 \rangle \Rightarrow \langle 1, 4, 4 \rangle \text{ just for simplicity}$$

The plane will look like  $x + 4y + 4z + d = 0$ , and we know it passes through  $(4, -2, 1)$

$$4 + 4(-2) + 4(1) + d = 0 \Rightarrow 4 - 8 + 4 + d = 0 \Rightarrow d = 0$$

$$x + 4y + 4z = 0$$



② Set up an integral for the surface area of  $S$ .

$$\iint_S dS = \iint_R |\vec{r}_u \times \vec{r}_v| dA$$

From part 1, we know  $\vec{r}_u \times \vec{r}_v = \langle 2u^2, 4uv, 2v^2 \rangle$ .

$$\text{So } |\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4 + 16u^2v^2 + 4v^4} = 2\sqrt{u^4 + 4u^2v^2 + v^4}$$

$$\text{S.A.} = \int_0^3 \int_{-3}^3 2\sqrt{u^4 + 4u^2v^2 + v^4} dv du$$

7) Is there a vector field  $\vec{G}$  on  $\mathbb{R}^3$  such that  $\text{curl } \vec{G} = \langle x, y, z \rangle$ ?

It is a fact (which you can easily check) that for any vector field  $\vec{F}$  with differentiable components,  $\text{div}(\text{curl}(\vec{F})) = 0$ .

$\text{div}(\text{curl}(\vec{G})) = \text{div}(\langle x, y, z \rangle) = 1 + 1 + 1 \neq 0$  so no such  $\vec{G}$  exists.

8) Find the work done by the force field  $\vec{F} = \langle z, x, y \rangle$  in moving a particle from the point  $(3, 0, 0)$  to the point  $(0, \pi/2, 3)$  along

① A straight line.

We need to parameterize the path.

$$\vec{r}(t) = \langle 3-3t, \frac{\pi}{2}t, 3t \rangle, \quad 0 \leq t \leq 1$$

$$\text{work} = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{r}' dt = \int_0^1 \langle 3t, 3-3t, \frac{\pi}{2}t \rangle \cdot \langle -3, \frac{\pi}{2}, 3 \rangle dt$$

$$= \int_0^1 -9t + \frac{3\pi}{2} - \frac{3\pi}{2}t + \frac{3\pi}{2}t dt = \int_0^1 \frac{3\pi}{2} - 9t dt = \frac{3\pi}{2}t - \frac{9t^2}{2} \Big|_0^1$$

$$= \frac{3\pi}{2} - \frac{9}{2} = \frac{1}{2}(3\pi - 9)$$

② The helix  $x = 3\cos t, y = t, z = 3\sin t$ .

$$\vec{r}(t) = \langle 3\cos t, t, 3\sin t \rangle, \quad 0 \leq t \leq \pi/2$$

$$\text{work} = \int_0^{\pi/2} \langle 3\sin t, 3\cos t, t \rangle \cdot \langle -3\sin t, 1, 3\cos t \rangle dt$$

$$= \int_0^{\pi/2} \underbrace{-9\sin^2 t}_{\sin^2 t = \frac{1-\cos 2t}{2}} + \underbrace{3\cos t + 3t\cos t}_{\text{I.B.P}} dt = \frac{-3\pi}{4}$$

9) Let  $\vec{F} = \langle x^2 - y^2, 2xy \rangle$  be the velocity field of a two-dimensional fluid flow. If  $D$  is the region in the first quadrant bounded by  $y = \sqrt{1-x^2}$ ,  $x=0$ ,  $y=0$  with its boundary  $\partial D$  oriented counterclockwise, find:

① the circulation of  $\vec{F}$  around  $\partial D$ .

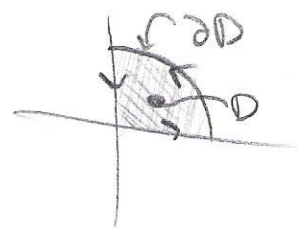
This is  $\int_{\partial D} \vec{F} \cdot d\vec{r}$ . Since  $\partial D$  is closed,

we use Green's theorem.

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left[ \frac{\partial}{\partial x} (2xy) - \frac{\partial}{\partial y} (x^2 - y^2) \right] dA = \iint_D 2y + 2y dA$$

$$= \int_0^{\pi/2} \int_0^1 4r^2 \sin \theta dr d\theta = \int_0^{\pi/2} \left. \frac{4r^3}{3} \right|_0^1 \sin \theta d\theta = \int_0^{\pi/2} \frac{4}{3} \sin \theta d\theta$$

$$= -\frac{4}{3} (\cos \theta \Big|_0^{\pi/2}) = -\frac{4}{3} (0 - 1) = \frac{4}{3}$$



② the flux of  $\vec{F}$  through  $\partial D$

This is  $\int_{\partial D} \vec{F} \cdot \vec{n} ds$ . Green's theorem can be used to show that this is equal to  $\iint_D \text{div} \vec{F} dA$ .

$$\int_{\partial D} \vec{F} \cdot \vec{n} ds = \iint_D \left[ \frac{\partial}{\partial x} (x^2 - y^2) + \frac{\partial}{\partial y} (2xy) \right] dA = \iint_D 4x dA$$

$$= \int_0^{\pi/2} \int_0^1 4r^2 \cos \theta dr d\theta = \int_0^{\pi/2} \left. \frac{4r^3}{3} \cos \theta \right|_0^1 d\theta = \frac{4}{3} (\sin \theta \Big|_0^{\pi/2})$$

$$= \frac{4}{3}$$

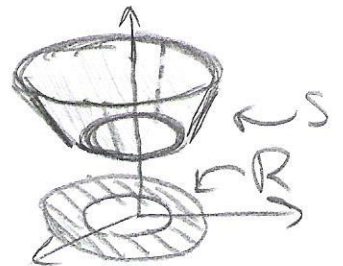
10) Compute the flux of  $\vec{F}$  across the given surface.

①  $\vec{F} = \langle \sin y, \sin z, yz \rangle$ ;  $S$  is the rectangular surface  $0 \leq y \leq 2$ ,  $0 \leq z \leq 3$  in the  $yz$ -plane with normal vector pointing in the negative  $x$ -direction.

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{N} dS &= \iint_R \langle \sin y, \sin z, yz \rangle \cdot \langle -1, 0, 0 \rangle dA \\ &= \int_0^2 \int_0^3 -\sin y dz dy = 3 \int_0^2 -\sin y dy = 3 (\cos y \Big|_0^2) = 3(\cos(2) - 1) \end{aligned}$$

②  $\vec{F} = \langle -x, -y, z^3 \rangle$ ;  $S$  is the part of the cone  $z = \sqrt{x^2 + y^2}$  between the planes  $z=1$  &  $z=3$  with downward orientation.

$$\iint_S \vec{F} \cdot \vec{N} dS = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA$$



We parameterize the surface by

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 1 \leq r \leq 3$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} = \langle r \cos \theta, r \sin \theta, -r \rangle$$

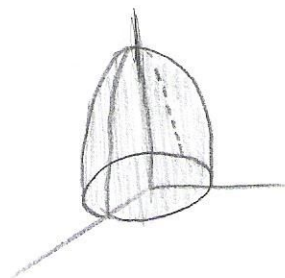
↑ this does point downward so no need to change it



$$\begin{aligned}
& \iint_R \langle -r \cos \theta, -r \sin \theta, r^3 \rangle \cdot \langle r \cos \theta, r \sin \theta, -r \rangle dA \\
&= \iint -r^2 \cos^2 \theta - r^2 \sin^2 \theta - r^4 dA = \iint -r^2 - r^4 dA \\
&= -\int_0^{2\pi} \int_0^3 r^4 + r^2 dr d\theta = -2\pi \left( \frac{r^5}{5} + \frac{r^3}{3} \right) \Big|_0^3 = -2\pi \left[ \left( \frac{243}{5} + \frac{27}{3} \right) - \left( \frac{1}{5} + \frac{1}{3} \right) \right] \\
&= -2\pi \left[ \frac{729+135-3-5}{15} \right] = -\frac{1712}{15} \pi
\end{aligned}$$

③  $\vec{F} = \langle 2x^3 + y^3, y^3 + z^3, 3y^2z \rangle$ ;  $S$  is the surface of the solid bounded by the paraboloid  $z = 1 - x^2 - y^2$  and the  $xy$ -plane.

Since  $S$  is closed, we can use Divergence Theorem.



$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} dS &= \iiint_V \operatorname{div} \vec{F} dV = \iiint_V 6x^2 + 3y^2 + 3y^2 dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} 6r^3 dz dr d\theta = 2\pi \int_0^1 6r^3(1-r^2) dr = 2\pi \int_0^1 6r^3 - 6r^5 dr \\
&= 2\pi \left( \frac{3r^4}{2} - r^6 \right) \Big|_0^1 = 2\pi \left( \frac{3}{2} - 1 \right) = \pi
\end{aligned}$$

11) True or false

① If  $\vec{F} = \langle P, Q \rangle$  and  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  in an open region  $D$ , then  $\vec{F}$  is conservative.

False.  $D$  has to be simply connected.

$$\textcircled{2} \int_{-c}^c f(x, y) ds = - \int_c^{-c} f(x, y) ds$$

False. A scalar line integral does not change when the path is reversed.

$$\textcircled{3} \int_{-c}^c \vec{F} \cdot d\vec{r} = - \int_c^{-c} \vec{F} \cdot d\vec{r}$$

True.

④ If  $S$  is a sphere and  $\vec{F}$  is a constant vector field,  $\iint_S \vec{F} \cdot d\vec{S} = 0$ .

True.  $S$  is closed, so use Div. Thm.  $\text{div} \vec{F} = 0$ .

⑤ The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve  $C$  is  $\oint_C y dx$ .

False. By Green's Theorem,  $\oint_C y dx = \iint_R \left[ \frac{\partial}{\partial x} [0] - \frac{\partial}{\partial y} [y] \right] dA = \iint_R -1 dA = -\text{area}$

⑥ The flux of  $\text{curl} \vec{F}$  through every oriented surface is zero.

False.  $\iint_S \text{curl} \vec{F} \cdot \vec{N} dS = 0$  is just not always true.

It is true if  $\vec{F}$  is conservative or if  $S$  is closed.