Review 4 : L23 - L30

1. Find the gradient vector field $\nabla\varphi$ of:

(a) $\varphi(x,y) = \tan^{-1}(y/x)$.

$$\nabla\varphi = \langle \varphi_x , \varphi_y \rangle$$

$$\varphi_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\varphi_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\nabla\varphi = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \frac{1}{x^2+y^2} \langle -y, x \rangle$$

$$= \frac{\langle -y, x\rangle}{\|r\|^2},$$

where $r = \langle x, y \rangle$

(b) $\varphi(x,y,z) = x\sqrt{y^2+z^2}$

$$\nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle =$$

$$= \left\langle \sqrt{y^2+z^2}, \frac{xy}{\sqrt{y^2+z^2}}, \frac{xz}{\sqrt{y^2+z^2}} \right\rangle.$$

2. (a) Find the gradient vector field F of $\varphi(x,y,z) = k\,(x^2+y^2+z^2)^{1/2}$ $(k>0)$

$$F = \nabla\varphi = \langle \varphi_x , \varphi_y, \varphi_z \rangle$$

$$= \left\langle k\frac{x}{(x^2+y^2+z^2)^{1/2}}, k\frac{y}{(x^2+y^2+z^2)^{1/2}}, k\frac{z}{(x^2+y^2+z^2)^{1/2}} \right\rangle$$

$$= \left\langle \frac{kx}{\|r\|}, \frac{ky}{\|r\|}, \frac{kz}{\|r\|} \right\rangle = \frac{k}{\|r\|}\langle x, y, z\rangle$$

$$= \frac{k}{\|r\|}\, r = k\frac{r}{\|r\|}$$

(b) Is F a radial vector field?

$$F = k\frac{r}{\|r\|}, \text{ where } r = \langle x, y, z\rangle \quad (k>0)$$

$\boxed{\text{Yes}}$ since $F(x,y,z) = k\frac{r}{\|r\|^p}$, where $p=1$.

(c) Is it directed inward towards the origin or outward from the origin?

$$F = \frac{k}{\|r\|} r \quad (k>0)$$

Since $\frac{k}{\|r\|} > 0$, F is directed outward from the origin.

(d) What is the magnitude of the vectors?

$$\|F\| = \frac{k}{\|r\|}\|r\| = \boxed{k}$$

3. Verify that a given vector field is conservative on the given domain D and find a potential function φ.

(a) $F = \langle \frac{1}{y}, -\frac{x}{y^2}\rangle$, where D is a simply connected domain in $\mathbb{R}^2$ that lies above the x-axis.

$$F = \langle F_1, F_2\rangle = \langle \frac{1}{y}, -\frac{x}{y^2}\rangle$$

$$\frac{\partial F_1}{\partial x} = 0; \quad \frac{\partial F_1}{\partial y} = -\frac{1}{y^2}; \quad \frac{\partial F_2}{\partial x} = -\frac{1}{y^2}; \quad \frac{\partial F_2}{\partial y} = \frac{2x}{y^3}.$$

First order partial derivatives of F_1 and F_2 are continuous on D and

$$\frac{\partial F_2}{\partial x} = -\frac{1}{y^2} = \frac{\partial F_1}{\partial y} \text{ on } D$$

$\Rightarrow$ F is conservative on D.

$\Rightarrow$ There exists a potential function $\varphi(x,y)$ on D such that

$$F = \nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle \frac{1}{y}, -\frac{x}{y^2} \rangle$$

$$\varphi_x = \frac{1}{y} \Rightarrow \varphi(x,y) = \int \frac{1}{y}\,dx = \frac{x}{y} + g(y)$$

$$\varphi_y = \frac{\partial}{\partial y}\left(\frac{x}{y} + g(y)\right) = -\frac{x}{y^2} + g'(y)$$

and

$$\varphi_y = -\frac{x}{y^2}$$

$$\Rightarrow -\frac{x}{y^2} + g'(y) = -\frac{x}{y^2}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C, \quad C \in R$$

$$\Rightarrow \boxed{\varphi(x,y) = \frac{x}{y} + C, \quad C \in R}$$

(b) $F = \langle F_1, F_2, F_3 \rangle = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 1 \rangle$ where $D = R^3$.

The first order partial derivatives of F_1, F_2, and F_3 are continuous on R^3, which is a simply connected domain, and

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \langle 6yz^2 - 6yz^2, 0 - 0, 2x - 2x \rangle$$

$$= \langle 0, 0, 0 \rangle$$

$\Rightarrow$ F is conservative on $D = R^3$.

$\Rightarrow$ There exists a potential function $\varphi(x,y,z)$ such that

$$F = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle F_1, F_2, F_3 \rangle$$

$$\frac{\partial \varphi}{\partial x} = 2xy \Rightarrow \varphi(x,y,z) = \int 2xy\,dx = x^2 y + g(y,z)$$

$$\frac{\partial \varphi}{\partial y} = x^2 + \frac{\partial}{\partial y} g(y,z) = x^2 + 2yz^3 \Rightarrow \frac{\partial g(y,z)}{\partial y} = 2yz^3$$

$$\Rightarrow g(y,z) = \int 2yz^3\,dy = y^2 z^3 + h(z)$$

$$\Rightarrow \varphi(x,y,z) = x^2 y + y^2 z^3 + h(z)$$

$$\frac{\partial \varphi}{\partial z} = 3y^2 z^2 + h'(z) = 3y^2 z^2 + 1 \Rightarrow h'(z) = 1$$

$$\Rightarrow h(z) = \int 1\,dz = z + C, \quad C \in \mathbb{R}$$

$$\Rightarrow \boxed{\varphi(x,y,z) = x^2 y + y^2 z^3 + z + C, \quad C \in \mathbb{R}}$$

(c) $F = \langle 2x\cos y, \; x^2 \sin y \rangle$, where $D = \mathbb{R}^2$

$F = \langle F_1, F_2 \rangle$

$$\frac{\partial F_2}{\partial x} = 2x \sin y$$

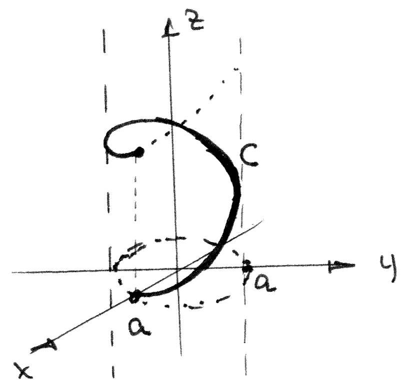
$$\frac{\partial F_1}{\partial y} = -2x \sin y$$

$$\frac{\partial F_2}{\partial x} \neq \frac{\partial F_1}{\partial y} \quad \text{on } D = \mathbb{R}^2$$

$\Rightarrow$ F is $\boxed{\text{not conservative}}$ on D.

4. Evaluate $\int_C (x^2 - y^2 + z)\,ds$, where the curve C is one turn of the helix

$r(t) = \langle a\cos t, \; a\sin t, \; bt \rangle$, $\quad 0 \le t \le 2\pi \; (a, b > 0)$



$$\int_C f(x,y,z)\,dS = \int_a^b f(r(t))\,\|r'(t)\|\,dt$$

$$r'(t) = \langle -a\sin t, \; a\cos t, \; b \rangle$$

$$\|r'(t)\| = \sqrt{a^2 + b^2}$$

$$\int_C (x^2-y^2+z)\,dS = \int_0^{2\pi} (a^2\cos^2 t - a^2\sin^2 t + bt)\sqrt{a^2+b^2}\,dt$$

$$= \sqrt{a^2+b^2}\int_0^{2\pi} \left[a^2(\cos^2 t - \sin^2 t) + bt\right]dt =$$

$$= \sqrt{a^2+b^2}\int_0^{2\pi} (a^2\cos 2t + bt)\,dt =$$

$$= \sqrt{a^2+b^2}\left(a^2 \frac{\sin 2t}{2}\Big]_0^{2\pi} + \frac{b}{2}t^2\Big]_0^{2\pi}\right) =$$

$$= \sqrt{a^2+b^2}\left(0 + \frac{b}{2}(2\pi)^2\right) = \boxed{2\pi^2 b\sqrt{a^2+b^2}}$$

5. Evaluate the work done by the force field $F = \langle x+yz, 2xy, z\rangle$ in moving a point object along the line segment from the point $(1,0,1)$ to the point $(2,3,1)$

Let $A = (1,0,1)$ and $B = (2,3,1)$

$v = AB = \langle 1,3,0\rangle$, $r_0 = \langle 1,0,1\rangle$

$C:\ r(t) = r_0 + tv = \langle 1+t, 3t, 1\rangle$, $0 \le t \le 1$

Work done by the force F:

$$W = \int_C F\cdot T\,ds = \int_C F\cdot dr = \int_0^1 F(r(t))\cdot r'(t)\,dt$$

$$F(r(t)) = \langle 1+t+3t,\ 2(1+t)3t,\ 1\rangle$$
$$= \langle 1+4t,\ 6+6t^2,\ 1\rangle$$

$$r'(t) = \langle 1, 3, 0\rangle$$

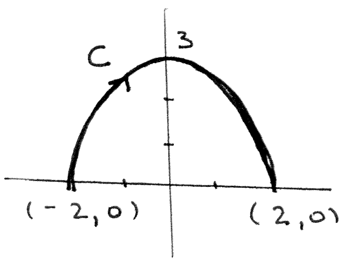
$$F(r(t))\cdot r'(t) = 1+4t+18t+18t^2 = 1+22t+18t^2$$

$$W = \int_C F\cdot dr = \int_0^1 (1+22t+18t^2)\,dt = t\Big]_0^1 + 11t^2\Big]_0^1 + 6t^3\Big]_0^1$$

$$= 1+11+6 = \boxed{18}$$

6. Evaluate the work done by the force $F = \langle y+1, x \rangle$ in moving a particle from the point $(-2,0)$ to the point $(2,0)$ along a path C:

(a) C is an upper half of an ellipse $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$ triversed clockwise.



Let $-C$ be the curve C with the reverse orientation. Then

$$-C: \begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases} \quad (0 \le t \le \pi)$$

or

$$-C: \quad r(t) = \langle 2\cos t, 3\sin t \rangle, \quad 0 \le t \le \pi$$

$$\text{Work} = W = \int_C F \cdot dr = -\int_{-C} F \cdot dr = -\int_0^{\pi} F(r(t)) \cdot r'(t)\, dt$$

$$F(r(t)) = \langle 3\sin t + 1, 2\cos t \rangle$$

$$r'(t) = \langle -2\sin t, 3\cos t \rangle$$

$$F(r(t)) \cdot r'(t) = -6\sin^2 t - 2\sin t + 6\cos^2 t$$

$$= 6(\cos^2 t - \sin^2 t) - 2\sin t = 6\cos 2t - 2\sin t$$

$$W = -\int_0^{\pi} F(r(t)) \cdot r'(t)\, dt = -\int_0^{\pi} (6\cos 2t - 2\sin t)\, dt$$

$$= -\left(6 \left.\frac{\sin 2t}{2}\right]_0^{\pi} + 2\cos t \Big]_0^{\pi} \right) = -2(\cos\pi - \cos 0)$$

$$= -2(-1-1) = \boxed{4}$$

(b) C is a line segment from $(-2,0)$ to $(2,0)$

Let $A = (-2,0)$ and $B = (2,0)$

$$V = AB = \langle 4, 0 \rangle, \quad r_0 = \langle -2, 0 \rangle$$

$$C: \quad r(t) = r_0 + tV = \langle -2+4t, 0 \rangle, \quad 0 \le t \le 1$$

$r'(t) = \langle 4, 0 \rangle$

$F(r(t)) = \langle 1, -2+4t \rangle$

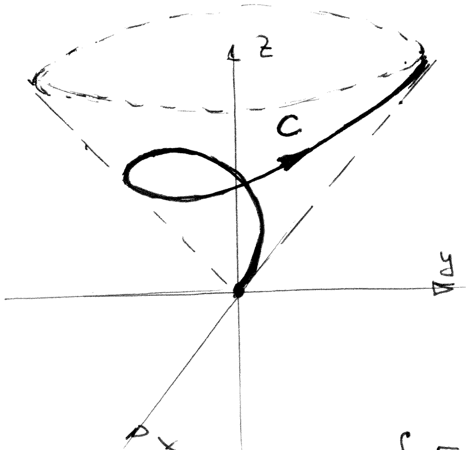
$F(r(t)) \cdot r'(t) = 4$

$$\text{Work} = \int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = \int_0^1 4\, dt = \boxed{4}.$$

The answers for two different paths are the same because $F = \langle y+1, x \rangle$ is a conservative vector field in $\mathbb{R}^2$.

7. The curve C is given by the parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ $(t \geq 0)$.

(a) Describe the curve:



$C: r(t) = \langle t\cos t, t\sin t, t \rangle$, $t \geq 0$

$x = t\cos t$, $y = t\sin t \Rightarrow x^2 + y^2 = t^2$

$\Rightarrow z = t = \sqrt{x^2+y^2}$

(b) Evaluate the line integral $\int_C F \cdot dr$, where $F(r) = r$ and $0 \leq t \leq 2\pi$.

$$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$F(\mathbf{r}) = r = \langle x, y, z \rangle$

$r(t) = \langle t\cos t, t\sin t, t \rangle$, $0 \leq t \leq 2\pi$

$F(r(t)) = \langle t\cos t, t\sin t, t \rangle$

$r'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 1 \rangle$

$F(r(t)) \cdot r'(t) = t\cos^2 t - t^2 \cos t \sin t + t\sin^2 t + t^2 \sin t \cos t + t$

$= t(\cos^2 t + \sin^2 t) + t = 2t$

$$\int_C F \cdot dr = \int_0^{2\pi} 2t\, dt = t^2 \Big|_0^{2\pi} = \boxed{4\pi^2}$$

8. (a) Show that a vector field $F = \langle x(1+y^2), x^2y \rangle$ is conservative in $\mathbb{R}^2$ and find the potential function φ.

$$F = \langle F_1, F_2 \rangle = \langle x + xy^2, x^2y \rangle$$

$$\frac{\partial F_2}{\partial x} = 2xy = \frac{\partial F_1}{\partial y} \quad \text{in } \mathbb{R}^2.$$

Also, the first order partial derivatives of F_1 and F_2 are continuous in a simply connected domain $D = \mathbb{R}^2$.

$\Rightarrow$ F is conservative in $\mathbb{R}^2$.

$\Rightarrow$ There exists a potential function φ such that

$$F = \nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle x + xy^2, x^2y \rangle$$

$$\varphi_y = x^2y \Rightarrow \varphi(x,y) = \int x^2y\,dy = \frac{x^2y^2}{2} + g(x)$$

$$\varphi_x = xy^2 + g'(x) = x + xy^2 \Rightarrow g'(x) = x$$

$$\Rightarrow g(x) = \frac{x^2}{2} + \mathbb{C}, \quad \mathbb{C} \in \mathbb{R}$$

$$\Rightarrow \varphi(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}x^2 + \mathbb{C}, \quad \mathbb{C} \in \mathbb{R}$$

or $\varphi(x,y) = \frac{x^2}{2}(y^2+1) + \mathbb{C}, \quad \mathbb{C} \in \mathbb{R}.$

(b) Evaluate $\int_C F \cdot dr$, where C is an arbitrary smooth curve from the point $(-1,3)$ to the point $(2,1)$.

$$\int_C F \cdot dr = \varphi(2,1) - \varphi(-1,3) = \frac{2^2}{2}(1+1) - \frac{1}{2}(3^2+1)$$

$$= 4 - 5 = \boxed{-1}$$

9. Given the vector field $F = \langle 2xy, x^2+2yz, y^2 \rangle$.

(a) Determine whether the vector field is conservative in $\mathbb{R}^3$.

$$F = \langle F_1, F_2, F_3 \rangle = \langle 2xy,\ x^2+2yz,\ y^2 \rangle$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \langle 2y-2y,\ 0-0,\ 2x-2x \rangle$$

$$= \langle 0, 0, 0 \rangle \quad \text{in } \mathbb{R}^3.$$

Also, the partial derivatives of F_1, F_2, F_3 are continuous in $\mathbb{R}^3$, which is a simply connected domain.

$\Rightarrow$ F is conservative in $\mathbb{R}^3$.

(b) Find the function φ such that $F = \nabla\varphi$

$$F = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle 2xy,\ x^2+2yz,\ y^2 \rangle$$

$$\varphi_x = 2xy \Rightarrow \varphi(x,y,z) = \int 2xy\,dx = x^2y + g(y,z)$$

$$\varphi_y = x^2 + g_y = x^2 + 2yz \Rightarrow g_y = 2yz$$

$$\Rightarrow g(y,z) = \int 2yz\,dy = y^2z + h(z)$$

$$\Rightarrow \varphi(x,y,z) = x^2y + y^2z + h(z)$$

$$\varphi_z = y^2 + h'(z) = y^2 \Rightarrow h'(z) = 0 \Rightarrow h(z) = C,\ C \in \mathbb{R}$$

$$\Rightarrow \boxed{\varphi(x,y,z) = x^2y + y^2z + C, \quad C \in \mathbb{R}}$$

(c) What is the value of the integral of the vector field F along any closed piecewise smooth curve in $\mathbb{R}^3$?

Since F is conservative in $\mathbb{R}^3$,

$$\oint_C F \cdot dr = 0$$

for any closed piecewise smooth curve in $\mathbb{R}^3$.

10. Consider the vector field $F = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$

(a) Show that the components of F have continuous partial derivatives on the domain $E = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\text{curl } F = 0$ on E.

$$F = \langle F_1, F_2 \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle.$$

$$\frac{\partial F_1}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

The partial derivatives are continuous on $E = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$$

$$= \langle 0, 0, 0 \rangle \quad \text{on } E.$$

(b) Evaluate $\oint_C F \cdot dr$, where C is a unit circle oriented counterclockwise. Is the vector field F conservative on E?

$$C: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad (0 \le t \le 2\pi)$$

$C: \; r(t) = \langle \cos t, \sin t \rangle \quad , \quad 0 \le t \le 2\pi$

$r'(t) = \langle -\sin t, \cos t \rangle$

$$F(r(t)) = \left\langle \frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle =$$

$$= \langle -\sin t, \cos t \rangle.$$

$$F(r(t)) \cdot r'(t) = \sin^2 t + \cos^2 t = 1$$

$$\oint_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

Since $\oint_C F \cdot dr \neq 0$, the vector field F $\boxed{\text{is not conservative on } E.}$

(c) Does (b) contradict to part (a)? Explain please.

$\boxed{\text{No}}$. Part (b) does not contradict to part (a) since E is not a simply connected domain.

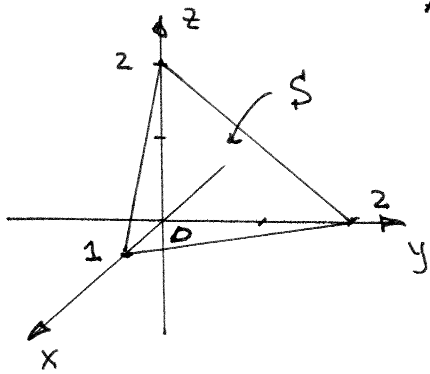
(d) What is the value of the integral $\oint_C F \cdot dr$ over a simple closed curve that neither passes through the origin nor encloses the origin? Justify please.

$\boxed{\oint_C F \cdot dr = 0}$ since there exists a simply connected domain containing C where F is conservative.

11. Determine the mass of the triangular plate whose vertices are $(1,0,0)$, $(0,2,0)$, $(0,0,2)$ if the mass density $\sigma(x,y,z) = xy$

$$\text{Mass} = \iint_S \sigma(x,y,z)\, dS = \iint_S xy\, dS$$

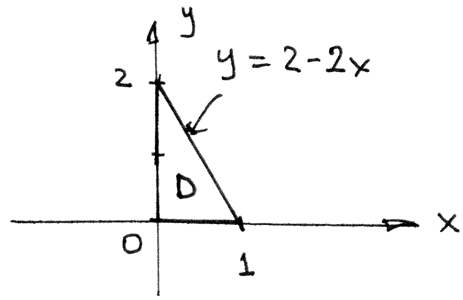
An equation of the plane containing S:



$$\frac{x}{1} + \frac{y}{2} + \frac{z}{2} = 1$$

$$2x + y + z = 2$$

$$z = 2 - 2x - y$$



$$S:\ z = 2 - 2x - y, \quad (x,y) \in D$$

$$D = \{0 \le x \le 1,\ 0 \le y \le 2 - 2x\}$$

$$dS = \|r_x \times r_y\|\, dx\, dy$$

$$r_x \times r_y = \langle -z_x, -z_y, 1\rangle = \langle 2, 1, 1\rangle$$

$$\|r_x \times r_y\| = \sqrt{6} \implies dS = \sqrt{6}\, dx\, dy$$

$$\text{Mass} = \iint_S xy\, dS = \iint_D xy\sqrt{6}\, dx\, dy = \sqrt{6}\int_0^1 x\, dx \int_0^{2-2x} y\, dy$$

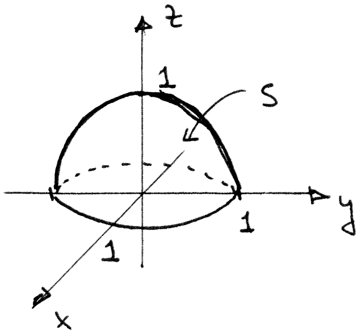
$$= \sqrt{6}\int_0^1 x\, dx\, \frac{y^2}{2}\Big|_0^{2-2x} = 2\sqrt{6}\int_0^1 x(1-x)^2 dx =$$

$$= 2\sqrt{6}\int_0^1 x(1-2x+x^2)\, dx = 2\sqrt{6}\int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= 2\sqrt{6}\left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right)\Big|_0^1 = 2\sqrt{6}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= 2\sqrt{6}\ \frac{6 - 8 + 3}{12} = \frac{2\sqrt{6}}{12} = \boxed{\frac{\sqrt{6}}{6}}$$

12. Find the surface area of the part of a paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane.



$S: z = 1 - x^2 - y^2, \quad (x,y) \in D$

$D: x^2 + y^2 \le 1$

Surface Area $= \iint_S 1\, dS$

$dS = \| r_x \times r_y \| \, dx\, dy$

$r_x \times r_y = \langle -z_x, -z_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$

$\| r_x \times r_y \| = \sqrt{4x^2 + 4y^2 + 1}.$

$$\iint_S 1\, dS = \iint_D \| r_x \times r_y \| \, dx\, dy = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right| = \int_0^{2\pi} d\theta \cdot \int_0^1 r\sqrt{4r^2+1}\, dr =$$

$$= 2\pi \int_0^1 r\sqrt{4r^2+1}\, dr = \frac{2\pi}{2\cdot 4} \int_0^1 \sqrt{4r^2+1}\, d(4r^2+1)$$

$\{$ substitution: $4r^2+1 = u \}$

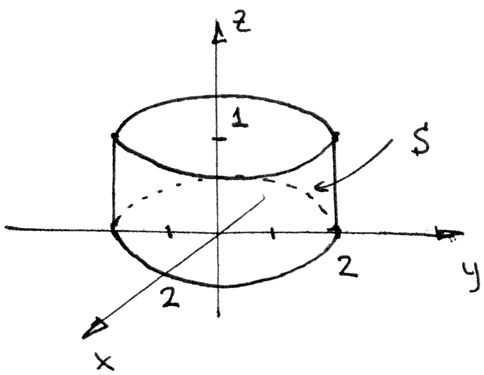
$$= \frac{\pi}{4} \cdot \frac{2}{3} (4r^2+1)^{3/2} \Big|_0^1 = \frac{\pi}{6}\left(5^{3/2} - 1\right) = \boxed{\frac{\pi}{6}\left(5\sqrt{5} - 1\right)}$$

13. Evaluate the surface integral $\iint_S x^2 dS$, where S is a part of a cylinder (top and bottom are not included): $x^2 + y^2 = 4, \quad 0 \le z \le 1.$

(a) Give a parameterization of the surface $r = r(u,v), \quad (u,v) \in D$

(b) Calculate $dS = \| r_u \times r_v \| \, du\, dv.$

(a) Using cylindrical coordinates:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

On the surface S:

$$\begin{cases} r = 2 \\ \theta = u \\ z = v \end{cases} \Rightarrow \begin{cases} x = 2\cos u \\ y = 2\sin u \\ z = v \end{cases}$$

$$\boxed{\begin{array}{l} S:\ r = r(u,v) = \langle 2\cos u,\ 2\sin u,\ v\rangle,\ (u,v)\in D \\ D = \{(u,v) \mid 0 \le u \le 2\pi,\ 0 \le v \le 1\} \end{array}}$$

(b) $r_u = \langle -2\sin u,\ 2\cos u,\ 0\rangle$

$r_v = \langle 0,\ 0,\ 1\rangle$

$r_u \times r_v = \langle 2\cos u,\ 2\sin u,\ 0\rangle$

$\| r_u \times r_v \| = \sqrt{4\cos^2 u + 4\sin^2 u} = 2$

$dS = \| r_u \times r_v \|\, du\, dv = \boxed{2\, du\, dv}$

(c) Compute $\iint_S x^2\, dS$.

$$\iint_S x^2\, dS = \iint_D (2\cos u)^2 \cdot 2\, du\, dv = 8 \iint_D \cos^2 u\, du\, dv$$

$$= 8 \int_0^{2\pi} \cos^2 u\, du \cdot \int_0^1 dv = 8 \int_0^{2\pi} \cos^2 u\, du =$$

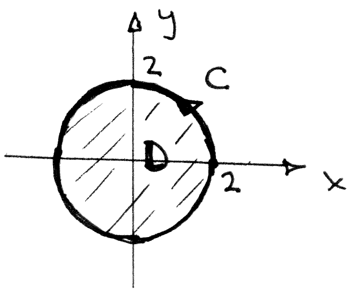
$$= 8 \cdot \frac{1}{2} \int_0^{2\pi} (1 + \cos 2u)\, du = 4 \left(u \Big]_0^{2\pi} + \frac{1}{2} \sin 2u \Big]_0^{2\pi} \right)$$

$$= 4\,(2\pi + 0) = \boxed{8\pi}$$

14. Evaluate the line integral by two methods: directly and by using Green's Theorem.

$$\int_C (x-y)\,dx + (x+y)\,dy,$$

where C is a circle with radius 2 and center at the origin oriented counterclockwise



1) Evaluating the integral directly:

C: $x = 2\cos t$, $y = 2\sin t$ $(0 \le t \le 2\pi)$

$$\int_C F_1\,dx + F_2\,dy =$$

$$= \int_0^{2\pi} \left[F_1(x(t),y(t))\,x'(t) + F_2(x(t),y(t))\,y'(t)\right] dt$$

$$\int_C (x-y)\,dx + (x+y)\,dy =$$

$$= \int_0^{2\pi} \left[(2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)\right] dt$$

$$= 4\int_0^{2\pi} (-\cos t\sin t + \sin^2 t + \cos^2 t + \sin t\cos t)\,dt =$$

$$= 4\int_0^{2\pi} 1\,dt = 4(2\pi) = \boxed{8\pi}$$

2) By Green's Theorem:

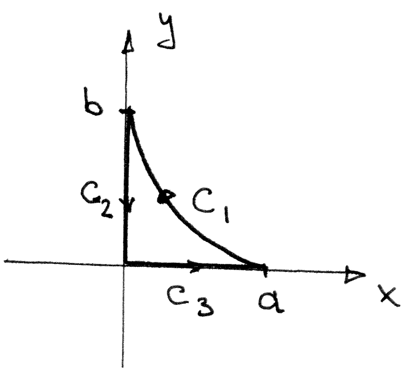
$$\int_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

D: $x^2 + y^2 \le 4$

$$\int_C (x-y)\,dx + (x+y)\,dy = \iint_D (1+1)\,dx\,dy =$$

$$= 2\iint_D 1\,dx\,dy = 2 \cdot \text{Area}(D) = 2\pi(2)^2 = \boxed{8\pi}$$

15. Use the line integral to find the area of the region in the first quadrant bounded by the astroid $\sqrt{\frac{|x|}{a}} + \sqrt{\frac{|y|}{b}} = 1$ $(a,b>0$ and the coordinate axes. (Hint: the astroid in the first quadrant can be parameterized as $r(t) = \langle a\cos^4 t, b\sin^4 t\rangle$, $0 \le t \le \frac{\pi}{2}$.



Let $C = C_1 \cup C_2 \cup C_3$

$$\text{Area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

C_1: $x = a\cos^4 t$, $y = b\sin^4 t$, $0 \le t \le \frac{\pi}{2}$.

$$\frac{1}{2}\int_{C_1} x\,dy - y\,dx =$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[a\cos^4 t \cdot 4b\sin^3 t\cos t - b\sin^4 t \cdot 4a\cos^3 t(-\sin t)\right] dt$$

$$= \frac{1}{2}\cdot 4ab\int_0^{\pi/2} \cos^3 t\sin^3 t\,(\cos^2 t + \sin^2 t)\,dt =$$

$$= 2ab\int_0^{\pi/2} \cos^3 t\sin^3 t\,dt = \frac{2ab}{8}\int_0^{\pi/2} \sin^3 2t\,dt =$$

$$= \frac{ab}{4}\cdot\frac{1}{2}\int_0^{\pi/2} (\sin^2 2t)\,d(-\cos 2t) =$$

$$= \frac{ab}{8}\int_0^{\pi/2} (\cos^2 2t - 1)\,d(\cos 2t) = \left|\begin{matrix}\text{substitution:}\\ \cos 2t = u\end{matrix}\right|$$

$$= \frac{ab}{8}\left(\frac{\cos^3 2t}{3}\Big|_0^{\pi/2} - \cos 2t\Big|_0^{\pi/2}\right) =$$

$$= \frac{ab}{8}\left[\frac{1}{3}(\cos^3\pi - \cos^3 0) - (\cos\pi - \cos 0)\right] =$$

$$= \frac{ab}{8}\left[\frac{1}{3}(-1-1) - (-1-1)\right] = \frac{ab}{8}\left(-\frac{2}{3}+2\right) =$$

$$= \frac{ab}{8}\ \frac{-2+6}{3} = \frac{ab}{\cancel{8}_2}\cdot\frac{\cancel{4}}{3} = \boxed{\frac{ab}{6}}$$

$-C_2: \quad x = 0, \quad y = t \qquad (0 \le t \le b)$

$$\frac{1}{2}\int_{C_2} x\,dy - y\,dx = -\frac{1}{2}\int_{-C_2} x\,dy - y\,dx =$$

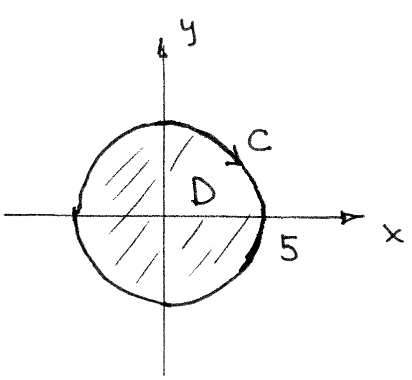
$$= -\frac{1}{2}\int_0^b (0\cdot 1 - t\cdot 0)\,dt = \boxed{0}$$

$C_3: \quad x = t, \quad y = 0 \qquad (0 \le t \le a)$

$$\frac{1}{2}\int_{C_3} x\,dy - y\,dx = \frac{1}{2}\int_0^a (t\cdot 0 - 0\cdot 1)\,dt = \boxed{0}$$

$$\Rightarrow \text{Area}(D) = \oint_C x\,dy - y\,dx = \frac{ab}{6} + 0 + 0 = \boxed{\frac{ab}{6}}$$

16. Use Green's Theorem to evaluate $\oint_C F\cdot dr$, where $F = \langle e^x + x^2, e^y - xy^2\rangle$ and C is the circle $x^2+y^2 = 25$ with clockwise orientation.



$$\oint_C F_1\,dx + F_2\,dy = -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$D: x^2+y^2 \le 25$;

Note: the negative sign in the formula above is needed due to the orientation of C.

$$F = \langle F_1, F_2 \rangle = \langle e^x + x^2 y,\ e^y - xy^2 \rangle$$

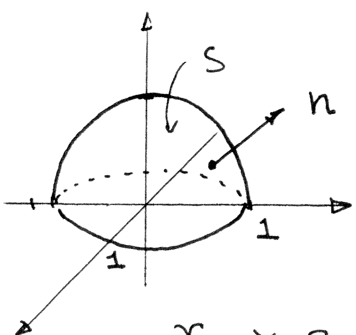
$$\frac{\partial F_2}{\partial x} = -y^2, \quad \frac{\partial F_1}{\partial y} = x^2$$

$$\oint_C F \cdot dr = -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = -\iint_D (-y^2 - x^2)\,dx\,dy$$

$$= \iint_D (x^2+y^2)\,dx\,dy = \left| \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \right| = \int_0^{2\pi} d\theta \int_0^5 r^3\,dr =$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_0^5 = \frac{\pi}{2}(5)^4 = \boxed{\frac{625\pi}{2}}$$

17. Find the flux of the vector field $F = \langle y, -x, z^2 \rangle$ across the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and oriented so that the z component of the normal vector is positive



$$S:\ z = 1 - x^2 - y^2, \quad (x,y) \in D$$

$$D:\ x^2 + y^2 \le 1, \quad z = 0$$

$$\text{Flux} = \iint_S F \cdot n\,dS = \iint_D F \cdot (r_x \times r_y)\,dx\,dy$$

$$r_x \times r_y = \langle -z_x, -z_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$$

$$F(r(x,y)) = \langle y, -x, (1-x^2-y^2)^2 \rangle.$$

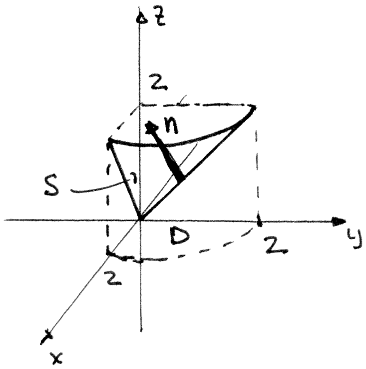
$$F \cdot (r_x \times r_y) = 2xy - 2xy + (1-x^2-y^2)^2 =$$
$$= (1-x^2-y^2)^2.$$

$$\text{Flux} = \iint_S F \cdot n\,dS = \iint_D (1-x^2-y^2)^2\,dx\,dy = \left| \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \right.$$

$$= \int_0^{2\pi} d\theta \int_0^1 r(1-r^2)^2\,dr = 2\pi \cdot \left(-\frac{1}{2}\right) \int_0^1 (1-r^2)^2\,d(1-r^2)$$

$$= \left\{ \begin{matrix} \text{substitution:} \\ u = 1 - r^2 \end{matrix} \right| = -\pi \frac{(1-r^2)^3}{3}\Big]_0^1 = \boxed{\frac{\pi}{3}}$$

18. Find the flux of the vector field $F = \langle xz, yz, 2z^2 \rangle$ across the part of the cone $z = \sqrt{x^2+y^2}$ that lies beneath the plane $z=2$ in the first octant and oriented so that the z-component of the normal vector is positive.



$S: z = \sqrt{x^2+y^2}, \quad (x,y) \in D$

$D: x^2+y^2 \le 4, \ x \geqslant 0, \ y \geqslant 0, \ z=0$

$$\text{Flux} = \iint_S F \cdot n \, dS = \iint_D F \cdot (r_x \times r_y)\, dx\, dy$$

$$r_x \times r_y = \langle -z_x, -z_y, 1 \rangle =$$

$$= \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

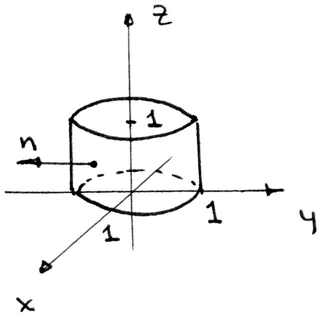
$$F(r(x,y)) = \langle x\sqrt{x^2+y^2}, \ y\sqrt{x^2+y^2}, \ 2(x^2+y^2) \rangle$$

$$F(r(x,y)) \cdot (r_x \times r_y) = -x^2 - y^2 + 2x^2 + 2y^2 = x^2+y^2.$$

$$\text{Flux} = \iint_S F \cdot n \, dS = \iint_D (x^2+y^2)\, dx\, dy = \int_0^{\pi/2} d\theta \int_0^2 r^3 dr =$$

$$= \frac{\pi}{2} \left. \frac{r^4}{4} \right|_0^2 = \frac{\pi}{2} \cdot \frac{16}{4} = \boxed{2\pi}$$

19. Evaluate the outward flux of the vector field $F = \langle xy^2, yz^2, zx^2 \rangle$ across the boundary of the solid enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$. (Hint: use the Divergence Theorem.)



By the Divergence Theorem:

$$\iint_S F\cdot n\, dS = \iiint_E \operatorname{div} F\, dV$$

$$E = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le 1\}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y^2 + z^2 + x^2.$$

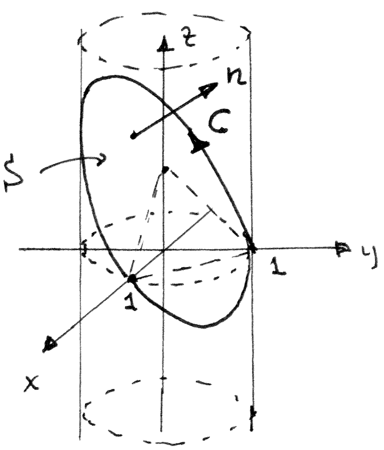
$$\iint_S F\cdot n\, dS = \iiint_E (x^2+y^2+z^2)\, dV = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_0^1 (r^2+z^2)\, dz =$$

$$\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right| = 2\pi \int_0^1 r\,dr \left(r^2 z + \frac{z^3}{3}\right)\Big|_0^1 =$$

$$= 2\pi \int_0^1 r\left(r^2 + \frac{1}{3}\right) dr = 2\pi \int_0^1 \left(r^3 + \frac{r}{3}\right) dr =$$

$$= 2\pi \left(\frac{r^4}{4} + \frac{r^2}{6}\right)\Big|_0^1 = 2\pi\left(\frac{1}{4} + \frac{1}{6}\right) = \pi\left(\frac{1}{2} + \frac{1}{3}\right) = \boxed{\frac{5\pi}{6}}$$

20. Evaluate the line integral of the vector field $F = \langle xy, yz, xz\rangle$ along the curve of intersection of the cylinder $x^2+y^2 = 1$ with the plane $x+y+z = 1$ if the curve is oriented counterclockwise as viewed from above. (Hint: use Stokes' Theorem).



By Stokes' Theorem:

$$\oint_C F\cdot dr = \iint_S (\operatorname{curl} F)\cdot dS = \iint_S (\operatorname{curl} F)\cdot n\, dS$$

$$\iint_S (\operatorname{curl} F)\cdot dS = \iint_D (\operatorname{curl} F)\cdot \langle r_x \times r_y\rangle\, dx\, dy$$

$S:\ z = 1 - x - y,\quad (x,y) \in D$

$D:\ x^2+y^2 \le 1,\quad z = 0$

$$r_x \times r_y = \langle -z_x, -z_y, 1\rangle = \langle 1, 1, 1\rangle$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = \langle -y, -z, -x\rangle$$

Evaluate curl F on the surface S:

$$(\text{curl } F)(r(x,y)) = \langle -y, -1+x+y, -x\rangle$$

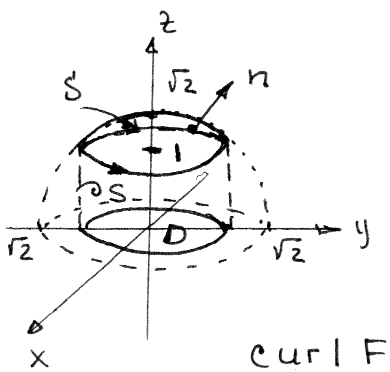
$$(\text{curl } F)\cdot(r_x \times r_y) = -y - 1 + x + y - x = -1$$

$$\oint_C F\cdot dr = \iint_S (\text{curl } F)\cdot n \, dS = \iint_D (\text{curl } F)\cdot(r_x \times r_y)\, dx\,dy$$

$$= \iint_D (-1)\, dx\,dy = -\text{Area}(D) = \boxed{-\pi}$$

21. Let $F = \langle y, -x, z\rangle$ and let S be the part of the sphere $x^2+y^2+z^2=2$ oriented upward that lies above the plane $z=1$. Evaluate:

(a) the flux of the curl F across the surface S;



$$\iint_S \text{curl } F\cdot n\, dS = \iint_D \text{curl } F\cdot(r_x \times r_y)\, dx\,dy$$

S: $z = \sqrt{2-x^2-y^2}$, $(x,y)\in D$

D: $x^2+y^2 \le 1$, $z=0$

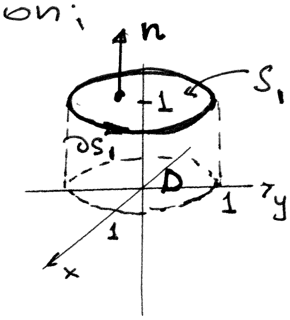
$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix} = \langle 0, 0, -2\rangle$$

$$r_x \times r_y = \langle -z_x, -z_y, 1\rangle = \left\langle \frac{x}{\sqrt{2-x^2-y^2}}, \frac{y}{\sqrt{2-x^2-y^2}}, 1\right\rangle$$

$$\text{curl } F\cdot(r_x \times r_y) = -2$$

$$\iint_S \text{curl } F\cdot n\, dS = \iint_D (-2)\, dx\,dy = (-2)(\text{Area}(D)) = \boxed{-2\pi}$$

(b) the flux of the curl F across the surface S_1, which is the projection of S onto the plane $z=1$, with upward orientation;



S_1: $z = 1$, $(x, y) \in D$

D: $x^2 + y^2 \leq 1$, $z = 0$

$r_x \times r_y = \langle -z_x, -z_y, 1 \rangle = \langle 0, 0, 1 \rangle$

$\text{curl} F = \langle 0, 0, -2 \rangle$

$\text{curl} F \cdot (r_x \times r_y) = -2$

$$\iint_{S_1} \text{curl} F \cdot n \, dS = \iint_D \text{curl} F \cdot (r_x \times r_y) \, dx\, dy =$$

$$= \iint_D (-2) \, dx\, dy = (-2)(\text{Area}(D)) = \boxed{-2\pi}$$

(c) the circulation of the vectos field F along the boundary of S, the curve ∂S (or, equivalently, across the boundary of S_1, which is the same curve ∂S) if the orientation of ∂S is consistent with the orientation of S (or S_1).

∂S: $r = r(t) = \langle \cos t, \sin t, 1 \rangle$, $0 \leq t \leq 2\pi$

$$\oint_{\partial S} F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt$$

$r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$F(r(t)) = \langle \sin t, -\cos t, 1 \rangle$

$F(r(t)) \cdot r'(t) = -\sin^2 t - \cos^2 t = -1$

$$\oint_{\partial S} F \cdot dr = \int_0^{2\pi} (-1) \, dt = \boxed{-2\pi}$$

(d) Stokes' Theorem explains why the answers in parts (a)-(c) are the same.

22. Which of the vector fields below are incompressible (or source free) on the given domain?

(a) $F = \langle xy, x-y^2, yz \rangle$ in $\mathbb{R}^3$.

$F = \langle F_1, F_2, F_3 \rangle$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y - 2y + y = 0 \quad \text{in } \mathbb{R}^3$$

$\Rightarrow$ $\boxed{F \text{ is incompressible in } \mathbb{R}^3.}$

(b) $F = \langle y^2+z^2, x^2+z^2, x^2+y^2)$ in $\mathbb{R}^3$.

$$\operatorname{div} F = 0 + 0 + 0 = 0 \quad \text{in } \mathbb{R}^3$$

$\Rightarrow$ $\boxed{F \text{ is incompressible in } \mathbb{R}^3}$

(c) $F = \dfrac{r}{\|r\|} = \left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$ in $\mathbb{R}^3 \setminus 0$.

$F = \langle F_1, F_2, F_3 \rangle$

$$\frac{\partial F_1}{\partial x} = \frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} = \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} =$$

$$= \frac{y^2+z^2}{\left[(x^2+y^2+z^2)^{1/2}\right]^3} = \frac{y^2+z^2}{\|r\|^3}$$

$$\frac{\partial F_2}{\partial y} = \frac{x^2+z^2}{\|r\|^3}; \quad \frac{\partial F_3}{\partial z} = \frac{x^2+y^2}{\|r\|^3}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{2(x^2+y^2+z^2)}{\|r\|^3} =$$

$$= \frac{2\|r\|^2}{\|r\|^3} = \boxed{\frac{2}{\|r\|}}$$

$\operatorname{div} F \neq 0$ on $\mathbb{R}^3 \setminus 0$.

$\Rightarrow$ $\boxed{F \text{ is not incompressible in } \mathbb{R}^3 \setminus 0}$

(d) $F = \langle a, b, c \rangle$ in $\mathbb{R}^3$ $(a, b, c \in \mathbb{R})$

$\operatorname{div} F = 0 \Rightarrow$ $\boxed{F \text{ is incompressible in } \mathbb{R}^3}$

23. Which of the vector fields are irrotational on the given domains?

(a) $F = \langle xy, x - y^2, yz \rangle$ in $\mathbb{R}^3$.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x-y^2 & yz \end{vmatrix} = \langle z, 0, 1-x \rangle$$

$\operatorname{curl} F \neq \langle 0, 0, 0 \rangle$ in $\mathbb{R}^3$ $\Rightarrow$ $\boxed{F \text{ is not irrotational in } \mathbb{R}^3.}$

(b) $F = \langle y^2 + z^2, x^2 + z^2, x^2 + y^2 \rangle$ in $\mathbb{R}^3$.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = 2\langle y - z, z - x, x - y \rangle$$

$\operatorname{curl} F \neq \langle 0, 0, 0 \rangle$ in $\mathbb{R}^3$

$\Rightarrow$ $\boxed{F \text{ is not irrotational in } \mathbb{R}^3}$

(c) $F = \frac{r}{\|r\|}$, $r = \langle x, y, z \rangle$ in $\mathbb{R}^3 \setminus 0$.

$$F = \langle F_1, F_2, F_3 \rangle = \left\langle \frac{x}{\|r\|}, \frac{y}{\|r\|}, \frac{z}{\|r\|} \right\rangle$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$r = \langle x, y, z \rangle, \quad \|r\| = \sqrt{x^2+y^2+z^2}.$$

$$F_1 = \frac{x}{\|r\|}, \quad F_2 = \frac{y}{\|r\|}, \quad F_3 = \frac{z}{\|r\|}.$$

$$\frac{\partial F_3}{\partial y} = z\left(-\frac{\frac{\partial}{\partial y}(\|r\|)}{\|r\|^2}\right) = -z\,\frac{\frac{y}{\|r\|}}{\|r\|^2} = -\frac{yz}{\|r\|^3}.$$

$$\frac{\partial F_2}{\partial z} = y\left(-\frac{\frac{\partial}{\partial z}(\|r\|)}{\|r\|^2}\right) = -y\,\frac{\frac{z}{\|r\|}}{\|r\|^2} = -\frac{yz}{\|r\|^3}.$$

$$\Rightarrow \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0$$

Similarly,

$$\frac{\partial F_1}{\partial z} = -\frac{xz}{\|r\|^3}, \quad \frac{\partial F_3}{\partial x} = -\frac{xz}{\|r\|^3}$$

$$\Rightarrow \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0$$

$$\frac{\partial F_2}{\partial x} = -\frac{xy}{\|r\|^3}, \quad \frac{\partial F_1}{\partial y} = -\frac{xy}{\|r\|^3}$$

$$\Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$\Rightarrow$ curl $F = \langle 0, 0, 0 \rangle$ in $\mathbb{R}^3 \setminus 0$

$\Rightarrow$ $\boxed{F \text{ is irrotational in } \mathbb{R}^3 \setminus 0}$

(d) $F = \langle a, b, c \rangle$ in $\mathbb{R}^3$ $(a, b, c \in \mathbb{R})$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$\Rightarrow$ $\boxed{F \text{ is irrotational in } \mathbb{R}^3}$.

24. Taking into consideration the symbolic notations:

curl $F = \nabla \times F$, div $F = \nabla \cdot F$, where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

determine which of the below (under certain conditions on the functions) are true, which are not true in general, and which do not make sense.

(a) $\nabla \cdot (\nabla \times F) = 0$ $\boxed{\text{True}}$

If $F = \langle F_1, F_2, F_3 \rangle$ and the components F_1, F_2, F_3 have continuous second order partial derivatives in a domain D, then

$$\nabla \cdot (\nabla \times F) = \text{div}\,(\text{curl}\, F) = 0$$

Indeed,

$$\text{curl}\, F = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$\text{div}\,(\text{curl}\, F) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0$$

(b) $\nabla \times (\nabla \cdot F) = 0$ $\boxed{\text{Does not make sense}}$

since $\nabla \cdot F = \text{div}\, F$ is a scalar function.

(c) $\nabla \cdot (\nabla \varphi) = 0$ $\boxed{\text{Not true in general}}$

Indeed, let $\varphi = \frac{1}{2}(x^2 + y^2 + z^2)$, then

$$\nabla \varphi = \langle x, y, z \rangle$$

$$\nabla \cdot (\nabla \varphi) = \text{div}\,(\nabla \varphi) = 1 + 1 + 1 = 3 \neq 0$$

(d) $\nabla \cdot (\nabla \varphi) = \nabla^2 \varphi$,

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. $\boxed{\text{True}}$

If $\varphi(x,y,z)$ has second order partial derivatives in a domain D, then

$$\nabla \cdot (\nabla \varphi) = \nabla^2 \varphi.$$

Indeed,

$$\nabla \cdot (\nabla \varphi) = \operatorname{div}(\nabla \varphi) = \operatorname{div}\left(\left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle\right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \nabla^2 \varphi$$

(e) $\nabla \times (\nabla \varphi) = \mathbf{0}$ $\boxed{\text{True}}$

If $\varphi(x,y,z)$ has continuous second order partial derivatives in a domain D, then

$$\nabla \times (\nabla \varphi) = \operatorname{curl}(\nabla \varphi) = 0$$

Indeed,

$$\nabla \varphi = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle$$

$$\operatorname{curl}(\nabla \varphi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \left\langle \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y}, \frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z}, \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right\rangle$$

$$= \langle 0, 0, 0 \rangle \quad \text{in } D.$$

(f) $\nabla \times (\nabla \times F) = 0$ $\boxed{\text{Not true in general}}$

For example, if $F = \langle x, y, xyz \rangle$, then

$$\nabla \times F = \text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & xyz \end{vmatrix} = \langle xz, -yz, 0 \rangle$$

$$\nabla \times (\nabla \times F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -yz & 0 \end{vmatrix} = \langle y, x, 0 \rangle$$

$\nabla \times (\nabla \times F) \neq \langle 0, 0, 0 \rangle$ in D.

(g) $\nabla \cdot (\nabla \cdot F) = 0$ $\boxed{\text{Does not make sense}}$

Since $\nabla \cdot F = \text{div } F$ is a scalar function

(h) $\nabla (\nabla \cdot F) = 0$ $\boxed{\text{Not true in general}}$

For example, if $F = \frac{1}{2} \langle x^2, y^2, z^2 \rangle$, then

$\nabla \cdot F = \text{div } F = x + y + z$

$\nabla (\nabla \cdot F) = \langle 1, 1, 1 \rangle \neq \langle 0, 0, 0 \rangle$