Review 4: L23 – L30

1. Find the gradient vector field $\nabla\varphi$ of:

(a) $\varphi(x,y)=\tan^{-1}(y/x)$; (b) $\varphi(x,y,z)=x\sqrt{y^2+z^2}$.

2. (a) Find the gradient vector field $\mathbf{F}$ of $\varphi(x,y,z)=k\left(x^2+y^2+z^2\right)^{1/2}$ $(k>0)$.

(b) Is $\mathbf{F}$ a radial vector field?
(c) Is it directed inward towards the origin or outward from the origin?
(d) What is the magnitude of the vectors?

3. Verify whether a given vector field is conservative on the given domain D and, if so, find a potential function φ.

(a) $\mathbf{F}=\left\langle \frac{1}{y},-\frac{x}{y^2}\right\rangle$, where D is a simply connected domain in $\mathbb{R}^2$ that lies above the x-axis.

(b) $\mathbf{F}=\left\langle 2xy,\, x^2+2yz^3,\, 3y^2z^2+1\right\rangle$, where $D=\mathbb{R}^3$.

(c) $\mathbf{F}=\left\langle 2x\cos(y),\, x^2\sin(y)\right\rangle$, where $D=\mathbb{R}^2$.

4. Evaluate $\int_C \left(x^2-y^2+z\right)ds$, where the curve C is one turn of the helix $\mathbf{r}(t)=\langle a\cos t,\, a\sin t,\, bt\rangle$, $0\le t\le 2\pi$ $(a,b>0)$.

5. Evaluate the work done by the force field $\mathbf{F}=\langle x+yz,\, 2xy,\, z\rangle$ in moving a point object along the line segment from the point $(1,0,1)$ to the point $(2,3,1)$.

6. Evaluate the work done by the force $\mathbf{F}=(y+1,\, x)$ in moving a particle from the point $(-2,0)$ to the point $(2,0)$ along a path C:

(a) C is an upper half of an ellipse $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$ traversed clockwise;

(b) C is a line segment from $(-2,0)$ to $(2,0)$.

Explain why the answers here for the two different parts are the same.

7. The curve C is given by the parametric equations $x=t\cos t,\ y=t\sin t,\ z=t$ $(t\ge 0)$.

   (a) Describe the curve.

   (b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{r})=\mathbf{r}$ and $0\le t\le 2\pi$.

8. (a) Show that the vector field $\mathbf{F}=\left\langle x\left(1+y^2\right),\, x^2y\right\rangle$ is conservative in $\mathbb{R}^2$ and find the potential function φ.

   (b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is an arbitrary smooth curve from the point $(-1,3)$ to the point $(2,1)$.

9. Given the vector field $\mathbf{F}=\left\langle 2xy,\, x^2+2yz,\, y^2\right\rangle$.

   (a) Determine whether the vector field is conservative in $\mathbb{R}^3$. If so,

(b) Find a function ϕ such that $\mathbf{F} = \nabla\phi$.
(c) What is the value of the integral of the vector field $\mathbf{F}$ along any closed piecewise smooth curve in $\mathbb{R}^3$?

10. Consider the vector field $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Show that the components of $\mathbf{F}$ have continuous partial derivatives on the domain $E = \mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbf{curl\ F} = \mathbf{0}$ on E.
(b) Evaluate $\oint_C \mathbf{F} \cdot \mathbf{dr}$, where C is a unit circle oriented counterclockwise. Is the vector field $\mathbf{F}$ conservative on E?
(c) Does (b) contradict to part (a)? Explain please.
(d) What is the value of the integral $\oint_C \mathbf{F} \cdot \mathbf{dr}$ over a simple closed curve that neither passes through the origin nor encloses the origin? Justify please.

11. Determine the mass of the triangular plate whose vertices are (1, 0, 0), (0, 2, 0), (0, 0, 2) if the mass density $\sigma(x, y, z) = xy$.

12. Find the surface area of the part of a paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane.

13. Evaluate the surface integral $\iint_S x^2 \, dS$, where S is the part of a cylinder (top and bottom are not included) : $x^2 + y^2 = 4, \ 0 \le z \le 1$.

(a) Give a parameterization of the surface $\mathbf{r} = \mathbf{r}(u,v), \ (u,v) \in D$

(b) Calculate $dS = |\mathbf{r}_u \times \mathbf{r}_v| dudv$.

(c) Compute $\iint_S x^2 \, dS$.

14. Evaluate the line integral by two methods: directly and by using Green's Theorem. $\int_C (x-y)dx + (x+y)dy$, where C is a circle with radius 2 and center at the origin oriented counterclockwise.

15. Use the line integral to find the area of the region in the first quadrant bounded by the astroid $\sqrt{\frac{|x|}{a}} + \sqrt{\frac{|y|}{b}} = 1 \quad (a, b > 0)$ and the coordinate axes. (Hint: the astroid in the first quadrant can be parameterized as $\mathbf{r}(t) = \langle a\cos^4 t, b\sin^4 t \rangle, \quad 0 \le t \le \pi/2$.)

16. Use Green's Theorem to evaluate $\oint_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = \langle e^x + x^2 y, e^y - xy^2 \rangle$ and C is the circle $x^2 + y^2 = 25$ with clockwise orientation.

17. Find the flux of the vector field $\mathbf{F} = \langle y, -x, z^2 \rangle$ across the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and oriented so that the z component of the normal vector is positive.

18. Find the flux of the vector field $\mathbf{F} = \langle xz, yz, 2z^2 \rangle$ across the part of the cone $z = \sqrt{x^2 + y^2}$ that lies beneath the plane $z = 2$ in the first octant and oriented so that the z component of the normal vector is positive.

19. Evaluate the outward flux of the vector field $\mathbf{F} = \langle xy^2, yz^2, zx^2 \rangle$ across the boundary of the solid enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$. (Hint: use the Divergence Theorem.)

20. Evaluate the line integral of the vector field $\mathbf{F} = \langle xy, yz, xz \rangle$ along the curve of intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$ if the curve is oriented counterclockwise as viewed from above. (Hint: use Stokes' Theorem)

21. Let $\mathbf{F} = \langle y, -x, z \rangle$ and let S be the part of the sphere $x^2 + y^2 + z^2 = 2$ oriented upward that lies above the plane $z = 1$. Evaluate directly the following:

(a) the flux of the **curl F** across the surface S;

(b) the flux of the **curl F** across the surface S_1, which is the projection of S onto the plane $z = 1$, with the upward orientation;

(c) the circulation of the vector field **F** along the boundary of *S*, the curve ∂S (or equivalently, across the boundary of S_1, which is the same curve ∂S) if the orientation of ∂S is consistent with the orientation of S (or S_1).

(d) Which theorem explains that answers obtained in parts (a) - (c) are the same?

22. Which of the vector fields below are incompressible (or source free) on the given domain?

(a) $\mathbf{F} = \langle xy, x - y^2, yz \rangle$ in $\mathbb{R}^3$

(b) $\mathbf{F} = \langle y^2 + z^2, x^2 + z^2, x^2 + y^2 \rangle$ in $\mathbb{R}^3$

(c) $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$, $\mathbf{r} = \langle x, y, z \rangle$ in $\mathbb{R}^3 \setminus \mathbf{0}$

(d) $\mathbf{F} = \langle a, b, c \rangle$ in $\mathbb{R}^3$ ($a, b, c \in \mathbb{R}$)

23. Which of the vector fields in question 22 are irrotational on the given domains?

24. Taking into consideration the symbolic notations:

$\mathbf{curl\,F} = \nabla \times \mathbf{F}$, div $\mathbf{F} = \nabla \cdot \mathbf{F}$, where $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$,

determine which of the below (under certain conditions on the functions) are true, which are not true in general, and which do not make sense:

(a) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(b) $\nabla \times (\nabla \cdot \mathbf{F}) = \mathbf{0}$

(c) $\nabla \cdot (\nabla \varphi) = 0$

(d) $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$, where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplace operator

(e) $\nabla \times (\nabla \phi) = \mathbf{0}$

(f) $\nabla \times (\nabla \times \mathbf{F}) = \mathbf{0}$

(g) $\nabla \cdot (\nabla \cdot \mathbf{F}) = 0$

(h) $\nabla (\nabla \cdot \mathbf{F}) = \mathbf{0}$