1. Convert the point $(1, -\sqrt{3}, -2\sqrt{3})$ from rectangular coordinates to

(1) Cylindrical coordinates

$r^2 = x^2 + y^2$ ; $x = r\cos\theta$ ; $y = r\sin\theta$ ; $\tan\theta = \frac{y}{x}$

$r^2 = (1)^2 + (-\sqrt{3})^2 = 1 + 3 = 4 \Rightarrow \boxed{r = 2}$

$1 = 2\cos\theta \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$ or $-\pi/3$

Since $y = -\sqrt{3}$ is negative, $\theta = -\pi/3$ is correct.

Typically, you give a positive $\theta$, so $\boxed{\theta = 5\pi/3}$

$(1, -\sqrt{3}, -2\sqrt{3}) \longrightarrow (2, 5\pi/3, -2\sqrt{3})$

(2) Spherical coordinates

$\rho^2 = x^2 + y^2 + z^2$ ; $x = \rho\sin\phi\cos\theta$ ; $y = \rho\sin\phi\sin\theta$ ; $z = \rho\cos\phi$

$\rho^2 = (1)^2 + (-\sqrt{3})^2 + (-2\sqrt{3})^2 = 4 + 12 = 16 \Rightarrow \boxed{\rho = 4}$

$\theta$ is the same as before. $\boxed{\theta = 5\pi/3}$

$-2\sqrt{3} = 4\cos\phi \Rightarrow \cos\phi = \frac{-\sqrt{3}}{2} \Rightarrow \phi = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

$\phi$ is conventionally between $0$ and $\pi$, so $\boxed{\phi = \frac{5\pi}{6}}$

$(1, -\sqrt{3}, -2\sqrt{3}) \longrightarrow \left(4, \frac{5\pi}{3}, \frac{5\pi}{6}\right)$

2. Identify the surface in cylindrical coordinates

(1) $r = 2\sin\theta$

$r^2 = 2r\sin\theta$

$x^2+y^2 = 2y$

$x^2+y^2-2y=0$

$x^2+y^2-2y+1=1$

$\boxed{x^2+(y-1)^2=1}$

cylinder

(2) $z=r^2\cos(2\theta)$

$z=r^2(\cos^2\theta-\sin^2\theta)$

$z=r^2\cos^2\theta-r^2\sin^2\theta$

$\boxed{z=x^2-y^2}$

hyperbolic paraboloid

3. Identify the surface in spherical coordinates

(1) $\rho = 4\cos\phi$

$\rho^2 = 4\rho\cos\phi$

$x^2+y^2+z^2=4z$

$x^2+y^2+z^2-4z=0$

$x^2+y^2+z^2-4z+4=4$

$\boxed{x^2+y^2+(z-2)^2=4}$

sphere

(2) $\cos^2\phi-\sin^2\phi=0$

$\cos^2\phi=\sin^2\phi$

$\rho^2\cos^2\phi=\rho^2\sin^2\phi$

$z^2=\rho^2\sin^2\phi\cdot 1$

$z^2=\rho^2\sin^2\phi\cdot(\sin^2\theta+\cos^2\theta)$

$z^2=\rho^2\sin^2\phi\sin^2\theta+\rho^2\sin^2\phi\cos^2\theta$

$\boxed{z^2=x^2+y^2}$

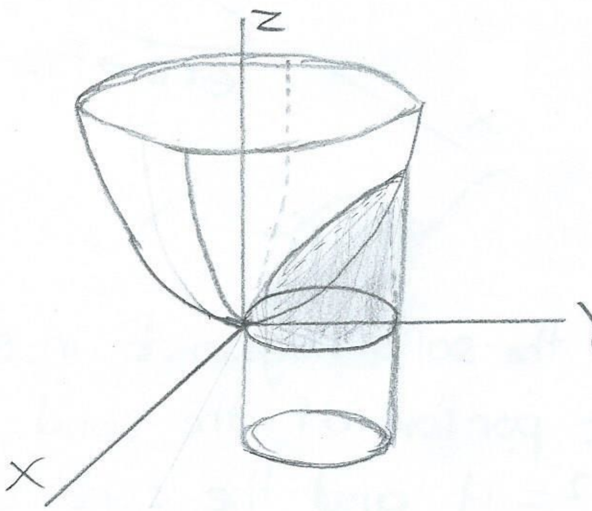
cone

4.

(1) Describe the solid region E in cylindrical coordinates if E is bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by $z=x^2+y^2$.

From problem 2.1, we know that cylinder is given by $r=2\sin\theta$. Also, the paraboloid is $z=r^2$.

Sketch of region:




We can see that the cylinder limits r and the paraboloid limits z.

$\theta$ is also determined by the cylinder. If you just test values for $\theta$, you'll see that the circle $r=2\sin\theta$ gets traced once as $\theta$ goes from 0 to $\pi$. This is generally true for $r=a\cdot\cos\theta$ or $r=a\cdot\sin\theta$.

We now have

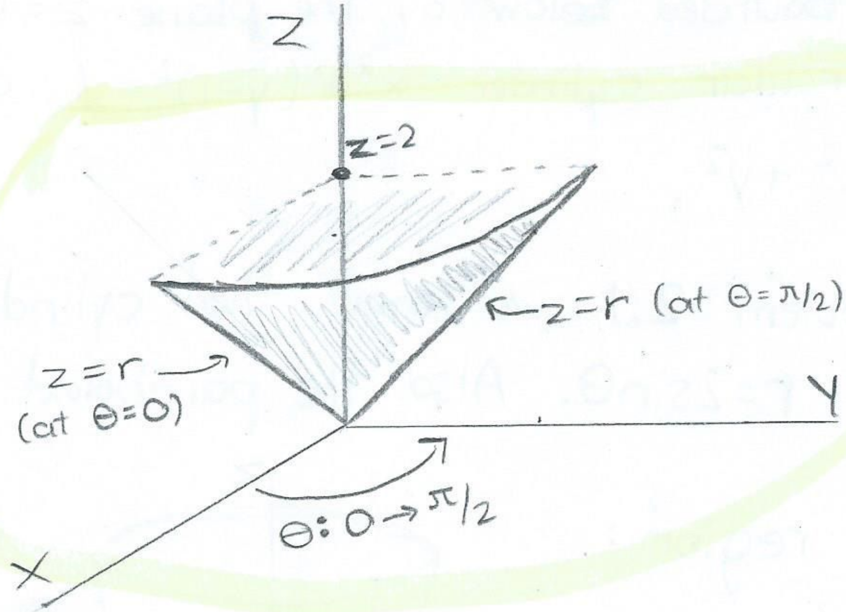
$0\le\theta\le\pi$

$0\le r\le 2\sin\theta$ which is compactly written as

$0\le z\le r^2$

$$\underbrace{E=\{(r,\theta,z)}_{\text{collection of triples}}\mid\underbrace{0\le\theta\le\pi,\ 0\le r\le 2\sin\theta,\ 0\le z\le r^2}_{\text{obeying these restrictions}}\}$$

(2) Sketch the solid $E=\{(r,\theta,z) \mid 0\le\theta\le\frac{\pi}{2},\ r\le z\le 2\}$.




5.

(1) Describe the solid region E in spherical coordinates if E is the portion of the solid bounded by the sphere $x^2+y^2+z^2=4$ and the cone $z^2=3(x^2+y^2)$ that lies in the first octant.

Similar to problem 3.2, the cone is $\cos^2\phi=3\sin^2\phi$. If we add $\sin^2\phi$ to both sides, we get $1=4\sin^2\phi$ which becomes $\sin\phi=\pm\frac{1}{2}$. Since we are in the first octant, $\phi=\frac{\pi}{6}$ is our upper limit on $\phi$. The sphere is clearly $\rho=2$. First octant means $\theta$ is between 0 and $\pi/2$

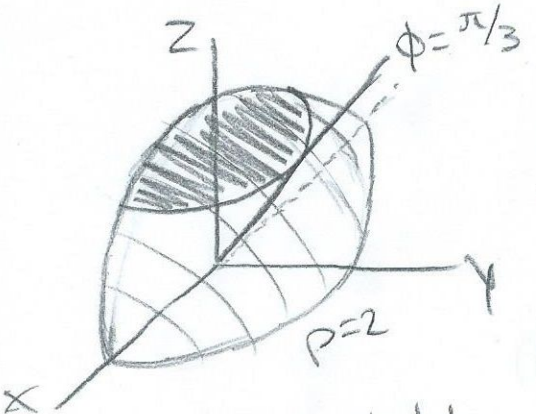
$$E=\{(\rho,\theta,\phi) \mid 0\le\rho\le2,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{6}\}$$

(2) Identify the solid $E = \{(\rho, \theta, \phi) \mid 0 \le \theta \le \pi,\ 0 \le \phi \le \pi/3,\ \frac{1}{\cos\phi} \le \rho \le 2\}$.

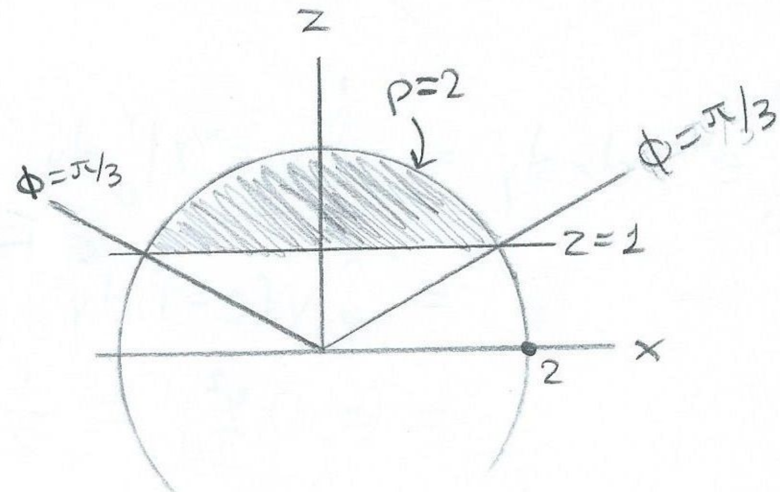
$\phi = \pi/3$ is the top half of a cone.

$\frac{1}{\cos\phi} \le \rho \rightarrow 1 \le \rho\cos\phi \rightarrow 1 \le z$ so we are above the plane $z=1$

$\rho \le 2$ means we are inside a sphere of radius 2




$\Longrightarrow$




very rough sketch

This type of shape is called a "cap" of the sphere $x^2+y^2+z^2=4$ with thickness 1. We are also looking only at the $y \ge 0$ half.

6. Evaluate the integrals

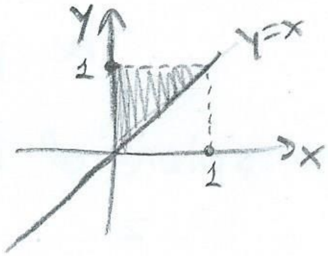
(1) $\int_0^4\int_0^5 \frac{1}{\sqrt{x+y}}\,dy\,dx = \int_0^4 2\sqrt{x+y}\Big|_0^5 dx = 2\int_0^4 \sqrt{x+5} - \sqrt{x}\,dx$

$= 2 \cdot \frac{2}{3}\left((x+5)^{3/2} - x^{3/2}\right)\Big|_0^4 = \frac{4}{3}\left[(9^{3/2} - 4^{3/2}) - (5^{3/2} + 0^{3/2})\right]$

$= \frac{4}{3}\left[27 - 8 - 5^{3/2}\right] = \frac{4}{3}(19 - 5\sqrt{5})$

(2) $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$

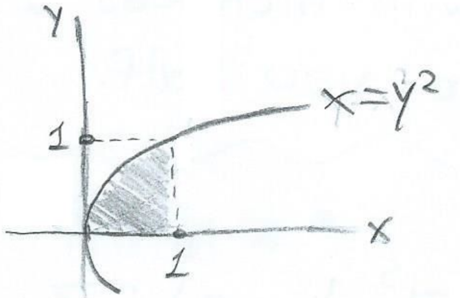
Impossible as is. We have to change order of integration.

$0 \le x \le 1$, $x \le y \le 1$ $\Rightarrow$  $\Rightarrow$ $0 \le y \le 1$, $0 \le x \le y$



$$\int_0^1 \int_0^y e^{x/y}\, dx\, dy = \int_0^1 y\, e^{x/y} \Big|_0^y dy$$
$$= \int_0^1 y(e-1)\, dy$$
$$= (e-1)\frac{y^2}{2}\Big|_0^1 = \frac{1}{2}(e-1)$$

(3) $\int_0^1 \int_{y^2}^1 y \sin(x^2)\, dx\, dy$ Change order of integration

$0 \le y \le 1$, $y^2 \le x \le 1$ $\Rightarrow$  $\Rightarrow$ $0 \le x \le 1$, $0 \le y \le \sqrt{x}$



$$\int_0^1 \int_0^{\sqrt{x}} y \sin(x^2)\, dy\, dx = \int_0^1 \sin(x^2)\frac{1}{2}y^2 \Big|_0^{\sqrt{x}} dy = \int_0^1 \frac{1}{2} x \sin(x^2)\, dx$$

$u = x^2$
$du = 2x\,dx$

$$= \frac{1}{4}\left(-\cos(x^2)\right)\Big|_0^1 = \frac{1}{4}\cos(x^2)\Big|_1^0 = \frac{1}{4}\left(1-\cos(1)\right)$$

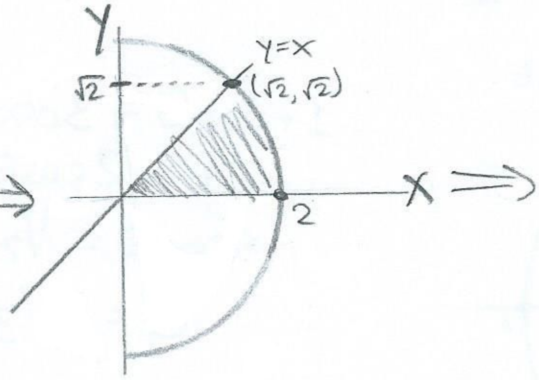
(4) $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$ Use polar coordinates

$0 \le y \le \sqrt{2}$

$y \le x \le \underbrace{\sqrt{4-y^2}}_{x=\sqrt{4-y^2}}$

right half of circle
$x^2+y^2=4$

$\Longrightarrow$




$\Longrightarrow$ $0 \le r \le 2$
$0 \le \theta \le \pi/4$

$$\int_0^{\pi/4}\int_0^2 \frac{r}{1+r^2}\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2}\ln(1+r^2)\Big|_0^2\,d\theta = \frac{1}{2}\int_0^{\pi/4} \ln(5)\,d\theta$$

$u = 1+r^2$
$du = 2r\,dr$

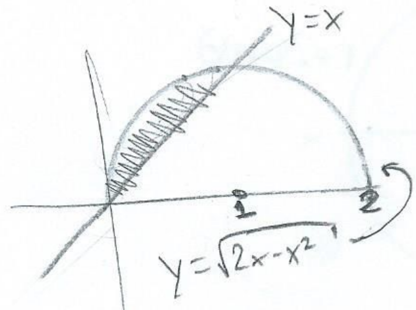
$= \boxed{\dfrac{\pi}{8}\ln(5)}$

7. Convert the integral $\displaystyle\int_0^1\int_x^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$ to polar coordinates.

$0 \le x \le 1$
$x \le y \le \sqrt{2x-x^2}$

$y = \sqrt{2x-x^2}$
$y^2 = 2x - x^2$
$x^2+y^2 = 2x$
$r^2 = 2r\cos\theta$
$r = 2\cos\theta$

$\Longrightarrow$

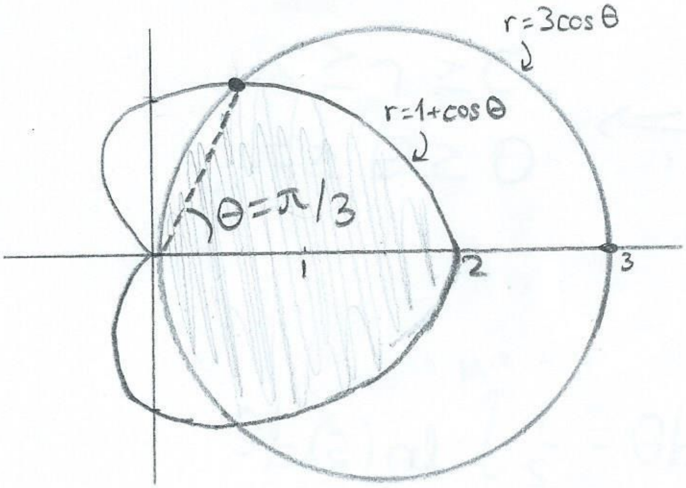



$\Longrightarrow$ $\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$
$0 \le r \le 2\cos\theta$

$$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r}\,r\,dr\,d\theta = \boxed{\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} dr\,d\theta}$$

8. Set up double integrals of the area of the region that

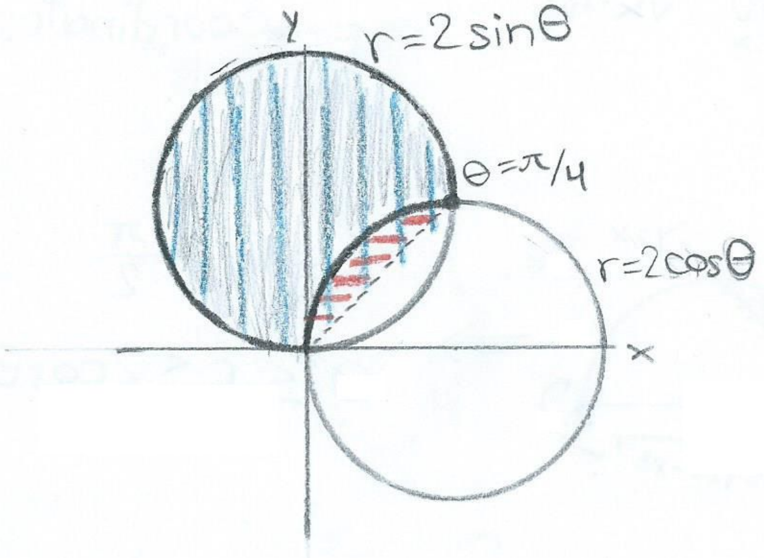
(1) lies inside both $r = 1 + \cos\theta$ and $r = 3\cos\theta$.




$$1 + \cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\cos\theta = 1/2$$
$$\theta = \pi/3$$

$$A = 2\left(\int_0^{\pi/3}\int_0^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{3\cos\theta} r\,dr\,d\theta\right)$$

(2) lies inside $r = 2\sin\theta$ and outside $r = 2\cos\theta$



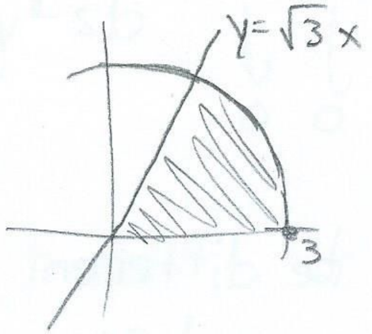

A= blue − red

$$A = \int_{\pi/4}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta - \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta$$

9. Express the following integrals in polar coordinates:

(1) $\iint_D (x^2+y^2)^{3/2}\,dA$ where D is the region in the first quadrant bounded by $y=0$, $y=\sqrt{3}x$, and $x^2+y^2=9$

Sketch:




$x^2+y^2=9 \rightarrow \boxed{r=3}$

$y=\sqrt{3}x \rightarrow r\sin\theta=\sqrt{3}r\cos\theta \rightarrow \tan\theta=\sqrt{3} \rightarrow \boxed{\theta=\pi/3}$

$$\int_0^{\pi/3}\int_0^3 (r^2)^{3/2}\,r\,dr\,d\theta = \int_0^{\pi/3}\int_0^3 r^4\,dr\,d\theta$$

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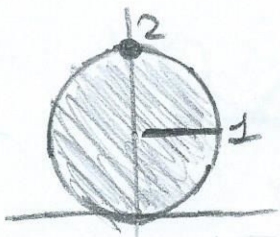
(2) $\iint_D \sqrt{x^2+y^2}\,dA$ where D is the closed disc of radius 1 centered at (0,1)

Sketch:

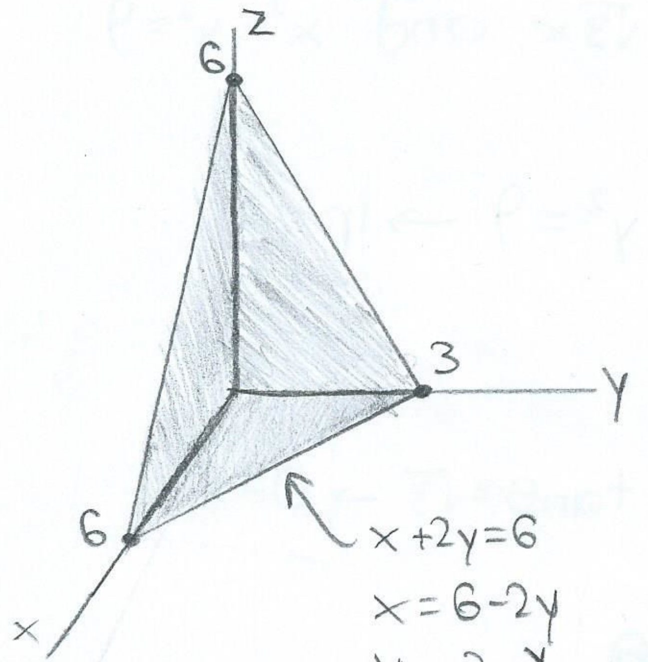



$x^2+(y-1)^2=1$

$r=2\sin\theta$

$$\int_0^{\pi}\int_0^{2\sin\theta} \sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta$$

10. Set up a triple integral for the volume of the solid in the first octant bounded by the coordinate planes and $z = 6 - x - 2y$.




$$V = \int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{6-x-2y} dz\,dy\,dx$$

will be different if you chose a different order of integration.

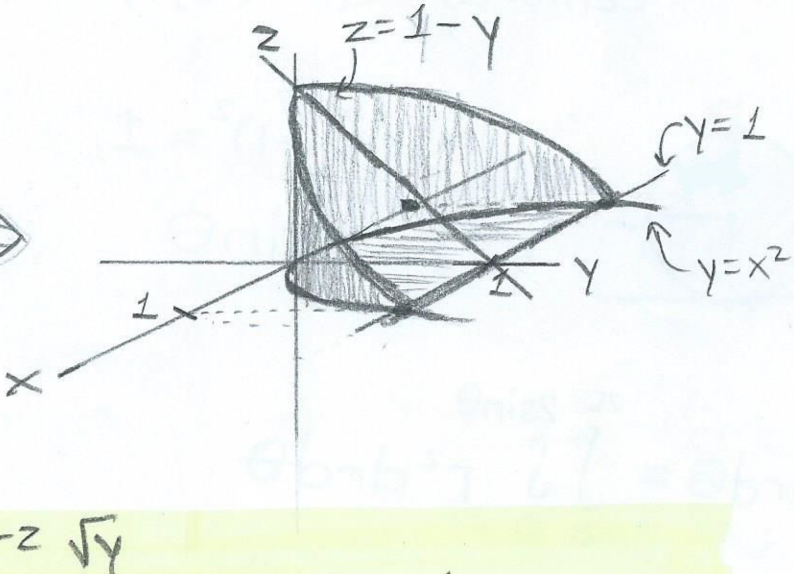
11. Rewrite $\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ in the order $dx\,dy\,dz$

$-1 \le x \le 1$

$x^2 \le y \le 1$

$0 \le z \le 1-y$

$\Rightarrow$




$0 \le z \le 1$

$0 \le y \le 1-z$

$-\sqrt{y} \le x \le \sqrt{y}$

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

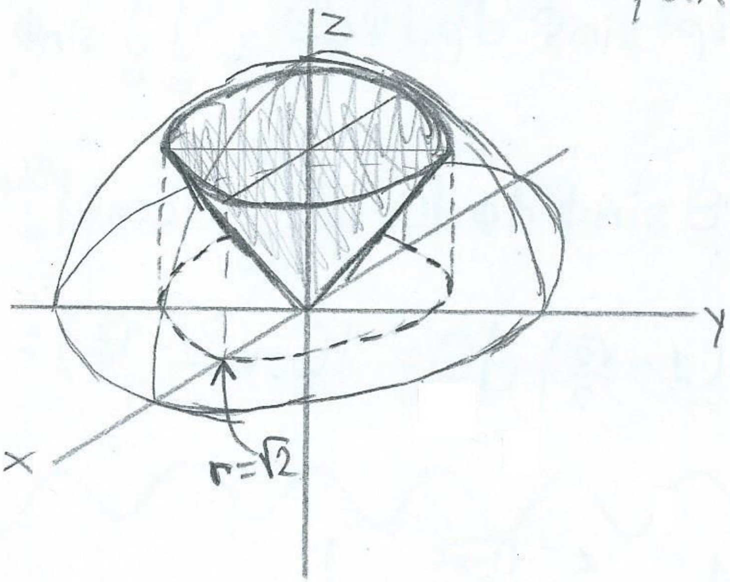
12. Convert $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3r\,dz\,dr\,d\theta$ to

(1) rectangular coordinates with order dzdydx

$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{2}$ $\Rightarrow$

$\underbrace{r \le}_{\text{cone}} \underbrace{z \le \sqrt{4-r^2}}_{\text{sphere}}$




$-\sqrt{2} \le x \le \sqrt{2}$

$-\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$

$\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}$

$$\boxed{\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\,dz\,dy\,dx}$$

(2) spherical coordinates

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

$0 \le \rho \le 2$

$z = r$ (cone)

$\rho\cos\phi = \rho\sin\phi$

$1 = \tan\phi$

$\phi = \pi/4$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

(3) Evaluate one of the integrals

Spherical is easiest.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \qquad \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \; \rho^3\Big|_0^2 \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} 8\sin\phi \, d\phi\, d\theta = 8\int_0^{2\pi} -\cos\phi\Big|_0^{\pi/4} d\theta = 8\int_0^{2\pi} \cos\phi\Big|_{\pi/4}^{0} d\theta$$

$$= 8\int_0^{2\pi}\left(1-\frac{\sqrt{2}}{2}\right) d\theta = 16\pi\left(1-\frac{\sqrt{2}}{2}\right) = 8\pi(2-\sqrt{2})$$

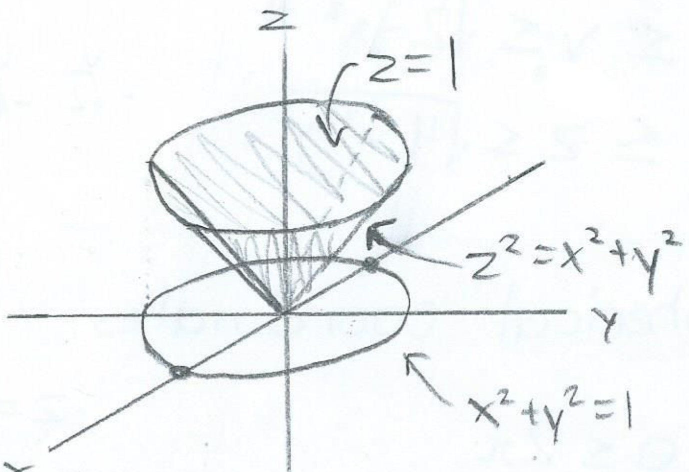
---

13. Convert $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dz\,dy\,dx$ to spherical and then evaluate

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$\sqrt{x^2+y^2} \le z \le 1 \quad \Longrightarrow$




$z = 1 \to \rho\cos\phi = 1 \to \rho = \sec\phi$

$z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$

$2z^2 = x^2+y^2+z^2$

$2\rho^2\cos^2\phi = \rho^2$

$2\cos^2\phi = 1$

$\cos\phi = \sqrt{2}/2$

$\phi = \pi/4$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

$0 \le \rho \le \sec\phi$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\sec^3\phi \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\sec^2\phi \tan\phi \, d\phi \, d\theta$$

$u = \tan\phi$

$du = \sec^2\phi \, d\phi$

$$= \frac{1}{3}\int_0^{2\pi} \frac{1}{2}\tan^2\phi \Big|_0^{\pi/4} d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \boxed{\frac{\pi}{3}}$$

14. Express $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}}\int_0^{x^2+y^2} f(x,y,z)\,dz\,dy\,dx$

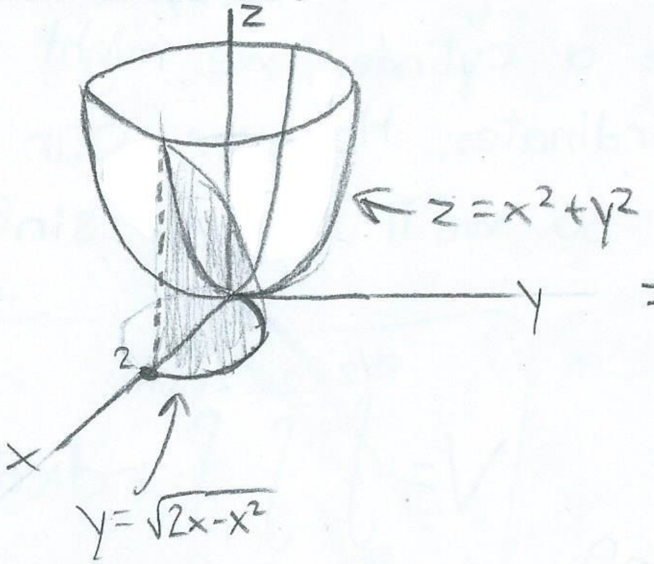
in cylindrical coordinates.

$0 \le x \le 2$

$0 \le y \le \sqrt{2x-x^2}$

$0 \le z \le x^2+y^2$

$\Longrightarrow$




$\Longrightarrow$

$0 \le \theta \le \pi/2$

$0 \le r \le 2\cos\theta$

$0 \le z \le r^2$

$$\boxed{\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

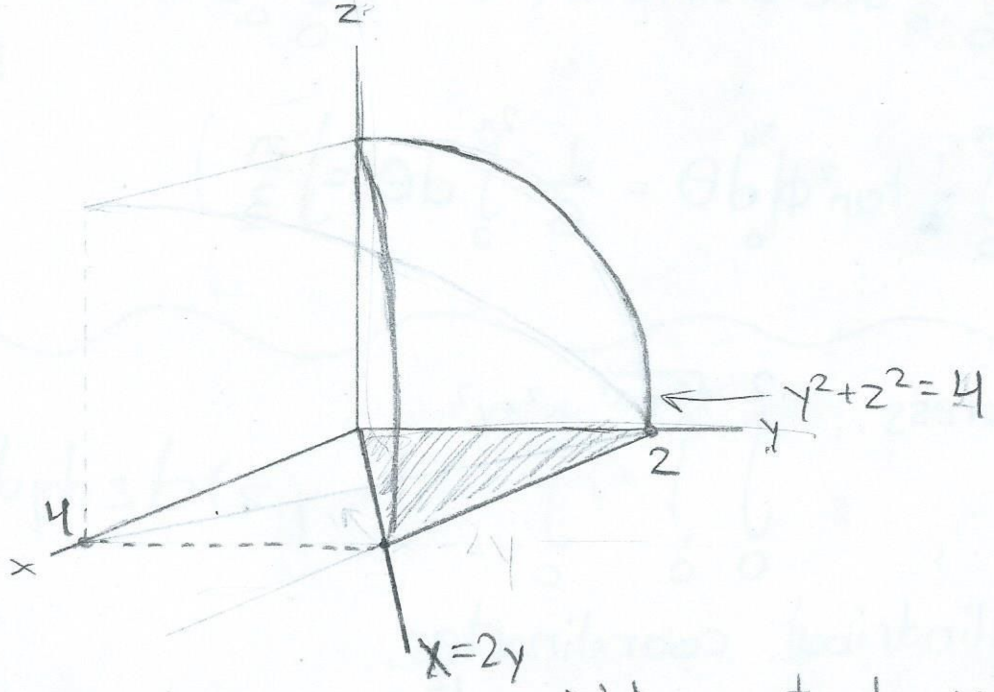
$y = \sqrt{2x-x^2}$

$x^2+y^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

15. Find the volume of the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant.




Since we have a cylinder, we might want to use cylindrical coordinates. However, our cylinder is in the yz-plane, so we'll say $y=r\sin\theta$ & $z=r\cos\theta$.

$0 \le \theta \le \pi/2$
$0 \le r \le 2$
$0 \le x \le 2r\sin\theta$

$$\boxed{V=\int_0^{\pi/2}\int_0^{2}\int_0^{2r\sin\theta} r\,dx\,dr\,d\theta}$$

we solve it on the next page

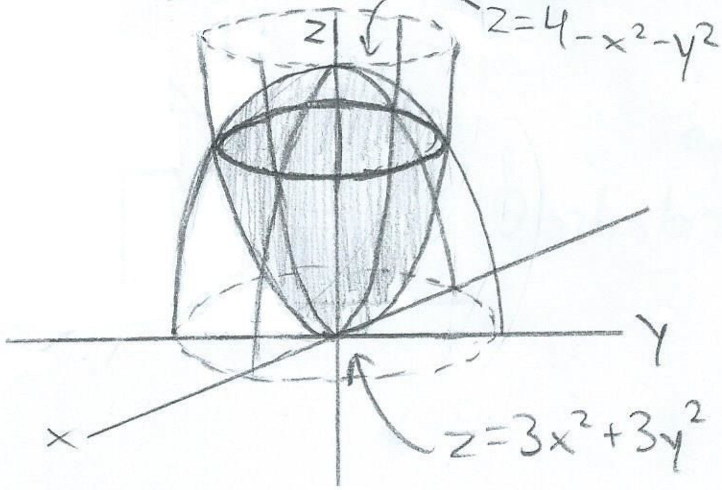
$$\int_0^{\pi/2}\int_0^2\int_0^{2r\sin\theta} r\,dx\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}r^3\Big|_0^2 \sin\theta\,d\theta = \int_0^{\pi/2}\frac{16}{3}\sin\theta\,d\theta = \frac{-16}{3}\cos\theta\Big|_0^{\pi/2}$$

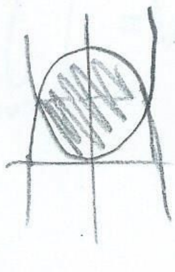
$$= \frac{-16}{3}(0-1) = \boxed{16/3}$$

16. Find the volume of the solid bounded by the paraboloids $z=3x^2+3y^2$ & $z=4-x^2-y^2$.

Sketch:




Rough sketch:



Use cylindrical coordinates. We'll need to find our $r\theta$-region. It's the circle of intersection.

$$4-x^2-y^2 = 3x^2+3y^2 \implies x^2+y^2=1$$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

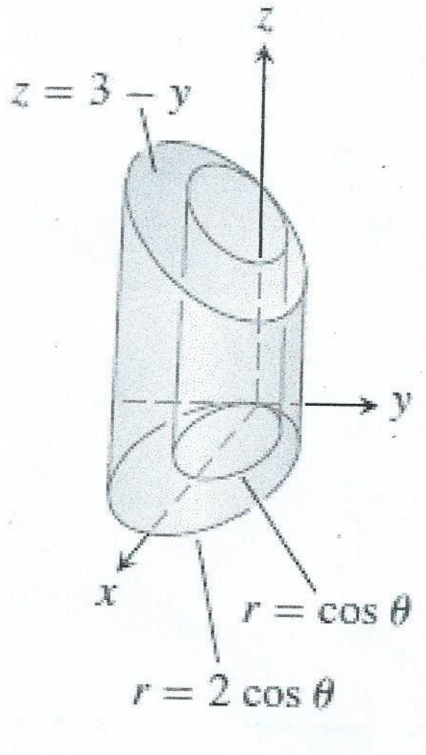
$3r^2 \le z \le 4-r^2$

$$V=\int_0^{2\pi}\int_0^1\int_{3r^2}^{4-r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 4r-4r^3\,dr\,d\theta$$

$$=\int_0^{2\pi} 2r^2-r^4\Big|_0^1\,d\theta=\int_0^{2\pi} 1\,d\theta=\boxed{2\pi}$$

17. Set up a triple integral for the volume of the solid whose base is the region between the circles $r=\cos\theta$ & $r=2\cos\theta$ and whose top lies in the plane $z=3-y$.

$$\boxed{V=\int_0^{\pi}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta}$$



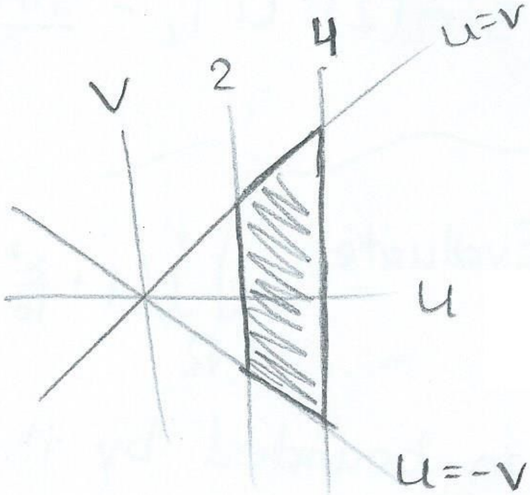
18. $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where $R$ is the region bounded by

$x+y=2$, $x+y=4$, $x=0$, $y=0$.

We will change the variables by saying $u=y+x$ and $v=y-x$. This becomes

$y=\frac{1}{2}(u+v)$ and $x=\frac{1}{2}(u-v)$.

$$\begin{array}{l} x+y=2 \\ x+y=4 \\ x=0 \\ y=0 \end{array} \Longrightarrow \begin{array}{l} u=2 \\ u=4 \\ u=v \\ u=-v \end{array}$$




$2 \le u \le 4$
$-u \le v \le u$

We also need the Jacobian.

$$J = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = 1/2$$

Now we have everything we need to solve the integral.

$$\int_2^4\int_{-u}^{u} \cos\left(\frac{v}{u}\right)\cdot\frac{1}{2}\,dv\,du = \frac{1}{2}\int_2^4 u\sin\left(\frac{v}{u}\right)\Big|_{-u}^{u}\,du$$

$$=\frac{1}{2}\int_2^4 u\left(\sin(1)-\sin(-1)\right)du = \frac{1}{2}\cdot 2\sin(1)\int_2^4 u\,du$$

$$=\frac{1}{2}\sin(1)\,u^2\Big|_2^4 = \frac{\sin(1)}{2}(16-4) = \boxed{6\sin(1)}$$

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19. Evaluate $\iint_R \left(1+\frac{x^2}{16}+\frac{y^2}{25}\right)^{3/2} dA$ where $R$ is the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$.

Change of coordinates again. If $R$ was a circle, we'd try polar coordinates. An ellipse is almost a circle, so we'll use almost-polar coordinates.

$x = 4r\cos\theta$

$y = 5r\sin\theta$

so $\frac{x^2}{16}+\frac{y^2}{25}=1 \Rightarrow \frac{16r^2\cos^2\theta}{16}+\frac{25r^2\sin^2\theta}{25}=1$

$\Rightarrow r=1$

$$\boxed{\begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{aligned}}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 4\cos\theta & -4r\sin\theta \\ 5\sin\theta & 5r\cos\theta \end{vmatrix}$$

$$= 20r\cos^2\theta + 20r\sin^2\theta = 20r$$

$$\int_0^{2\pi}\int_0^1 (1+r^2)^{3/2} \cdot 20r\, dr\, d\theta = \int_0^{2\pi}\int_1^2 10\, u^{3/2}\, du\, d\theta$$

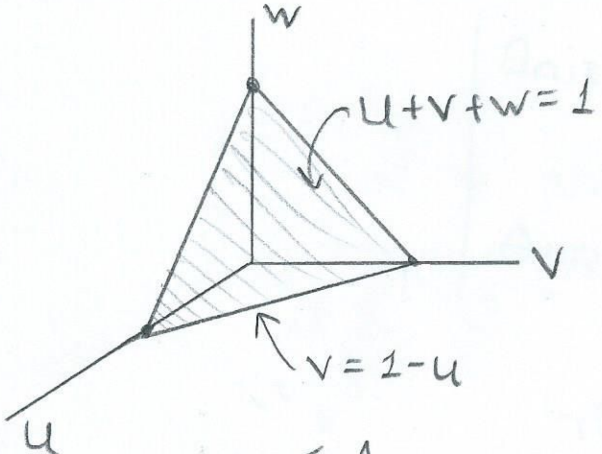
$u = 1+r^2$ $\quad u_t = 1+1^2 = 2$
$du = 2r\,dr$ $\quad u_b = 1+0^2 = 1$

$$= \int_0^{2\pi} 10 \cdot \frac{2}{5} u^{5/2} \Big|_1^2 d\theta = \int_0^{2\pi} 4(4\sqrt{2}-1)\, d\theta = 8\pi(4\sqrt{2}-1)$$

20. Use the transformation $x=u^2$, $y=v^2$, $z=w^2$ to set up an integral for the volume of the region bounded by $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and the coordinate planes

$\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ $\qquad$ $u+v+w=1$

$x=0$ $\qquad \Longrightarrow \qquad$ $u=0$
$y=0$ $\qquad\qquad\qquad$ $v=0$
$z=0$ $\qquad\qquad\qquad$ $w=0$




$0 \le u \le 1$

$0 \le v \le 1-u$

$0 \le w \le 1-u-v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix}$$

$$= 2u \begin{vmatrix} 2v & 0 \\ 0 & 2w \end{vmatrix} = 8uvw$$

$$\boxed{V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dw\,dv\,du}$$

21.

(1) For any region D in the plane, $\iint_D dA \ge 0$.

True. That integral is the area of D, which can't be negative.

(2) For any region $D$ in the plane, $\iint_D f(x,y)\,dA \geq 0$.

False. For example, if $f(x,y) = -1$, the integral will be negative.

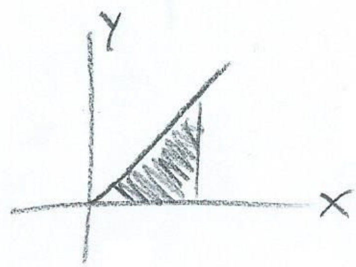
(3) If $f$ is continuous on $[a,b]\times[c,d]$, then
$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dy\,dx.$$

False. If you flip the bounds, you have to flip the differentials as well.

(4) $$\int_0^1\int_0^x f(x,y)\,dy\,dx = \int_0^1\int_0^y f(x,y)\,dx\,dy$$

False.

$0 \leq x \leq 1$
$0 \leq y \leq x$ $\Rightarrow$



The regions are different, so the integrals might be different.

$0 \leq y \leq 1$
$0 \leq x \leq y$ $\Rightarrow$

(5) If the point P is on the surface $\phi=0$, then P lies in the xy-plane.

False. If $\phi=0$, then
$$\begin{aligned} x&=\rho\sin\phi\cos\theta \\ y&=\rho\sin\phi\sin\theta \\ z&=\rho\cos\phi \end{aligned} \quad \text{becomes} \quad \begin{aligned} x&=0 \\ y&=0 \\ z&=\rho \end{aligned}$$

So $\phi=0$ is the positive z-axis. But the xy-plane is $z=0$. For example, $(0,0,5)$ is on $\phi=0$ but not the xy-plane.

(6) If P is on the surface $\theta=0$, then P lies in the xz-plane.

Like above, $\theta=0$ leads to
$$\begin{aligned} x&=\rho\sin\phi \\ y&=0 \\ z&=\rho\cos\phi \end{aligned}$$

The xz-plane is the plane $y=0$. So yes, the statement is true.