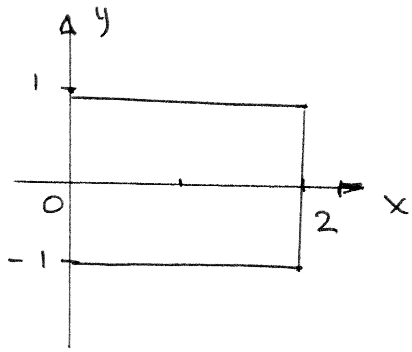
Review 3

3. Consider the function $f(x,y) = x + y^2$ on the rectangle $R = \{(x,y) \mid 0 \le x \le 2, -1 \le y \le 1\}$.
Evaluate $\iint_R f(x,y)\,dA$.



$$\iint_R (x+y^2)\,dA = \int_{-1}^{1} dy \int_0^2 (x+y^2)\,dx$$

$$= \int_{-1}^{1} dy \left(\frac{x^2}{2} + xy^2 \right)\Big]_0^2 =$$

$$= \int_{-1}^{1} (2 + 2y^2)\,dy = 2\int_{-1}^{1} (1+y^2)\,dy$$

$$= 2\left(y + \frac{y^3}{3}\right)\Big]_{-1}^{1} = 2\left(2 + \frac{2}{3}\right) = 2 \cdot \frac{8}{3} = \boxed{\frac{16}{3}}$$

4. Find the volume of a solid beneath the plane $f(x,y) = 3x + 3y + 9$ and above the region $R = \{(x,y) \mid -1 \le x \le 1,\ 1 \le y \le 3\}$.

$$V = \iint_R f(x,y)\,dA = \int_1^3 dy \int_{-1}^{1} (3x + 3y + 9)\,dx =$$

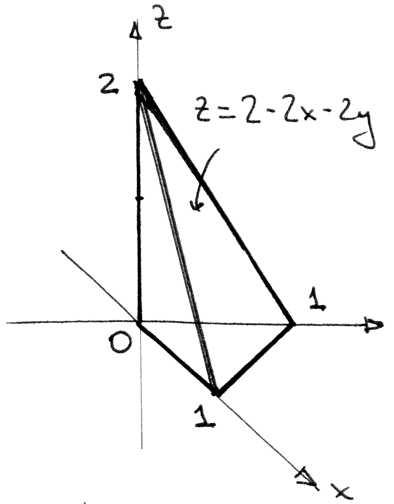
$$= \int_1^3 dy \cdot \int_{-1}^{1} 3x\,dx + \int_1^3 (3y+9)\,dy \cdot \int_{-1}^{1} dx =$$

$$= 0 + \left(\frac{3}{2} y^2 \Big]_1^3 + 9y\Big]_1^3 \right) \cdot x\Big]_{-1}^{1} = \left(\frac{3}{2}(8) + 9(2) \right) \cdot 2$$

$$= 2(12 + 18) = \boxed{60}$$

5. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 2$. Do it in three ways:
(1) using a formula from Geometry;
(2) using a double integral; (3) using a triple integral.

$$2x + 2y + z = 2 \Rightarrow x + y + \frac{z}{2} = 1$$



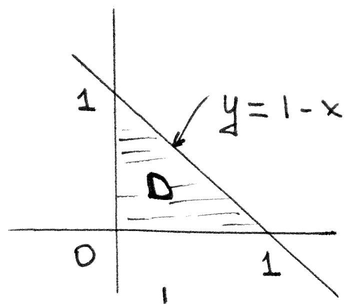
(1) The volume of the tetrahedron:

$$V = \frac{1}{3}\,[\text{Area of Base}] \times [\text{Height}]$$

$$V = \frac{1}{3}\left[\frac{1}{2}\cdot 1\cdot 1\right] \times 2 = \boxed{\frac{1}{3}}$$

(2) $2x + 2y + z = 2$

$$z = 2 - 2x - 2y$$



$$V = \iint_D (2 - 2x - 2y)\, dA$$

$$= \int_0^1 dx \int_0^{1-x} (2 - 2x - 2y)\, dy =$$

$$= \int_0^1 (2-2x)\, dx \int_0^{1-x} dy - \int_0^1 dx \int_0^{1-x} 2y\, dy =$$

$$= 2\int_0^1 (1-x)^2 dx - \int_0^1 dx\; y^2 \Big|_0^{1-x} = 2\int_0^1 (1-x)^2 dx - \int_0^1 (1-x)^2 dx$$

$$= \int_0^1 (1-x)^2 dx = -\frac{(1-x)^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$$

(3) $$V = \int_0^1 dx \int_0^{1-x} dy \int_0^{2-2x-2y} dz = \int_0^1 dx \int_0^{1-x} (2-2x-2y)\, dy = \boxed{\frac{1}{3}}$$

(see part 2)

6. Given an iterated integral:

$$\int_0^1 \int_0^3 4xy\, e^{x^2} dx\, dy = \int_0^1 2y\, dy \int_0^3 2x e^{x^2} dx.$$

(a) Rewrite it as a product of two single integrals if possible.

$$\int_0^1 2y\, dy \int_0^3 2xe^{x^2} dx = \int_0^1 2y\, dy \cdot \int_0^3 2xe^{x^2} dx$$

(b) Evaluate the integral.

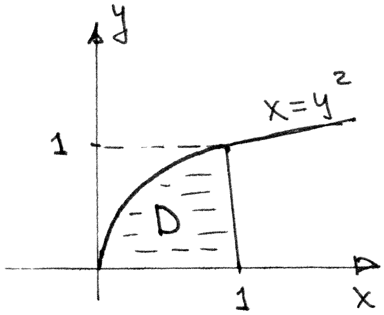
$$\int_0^1 2y\,dy \cdot \int_0^3 2xe^{x^2}dx = y\Big]_0^1 \cdot e^{x^2}\Big]_0^3 = e^9 - 1$$

7. Given an iterated integral: $\int_0^1 x\,dx \int_0^{\sqrt{x}} \sqrt{y}\,dy$.

(a) Evaluate the integral.

$$\int_0^1 x\,dx \int_0^{\sqrt{x}} \sqrt{y}\,dy = \int_0^1 x\,dx\ \frac{2}{3}y^{3/2}\Big]_0^{\sqrt{x}} = \frac{2}{3}\int_0^1 x\left(x^{\frac{1}{2}}\right)^{3/2}dx$$

$$= \frac{2}{3}\int_0^1 x^{7/4}dx = \frac{2}{3}\cdot\frac{4}{11}\,x^{11/4}\Big]_0^1 = \boxed{\frac{8}{33}}$$



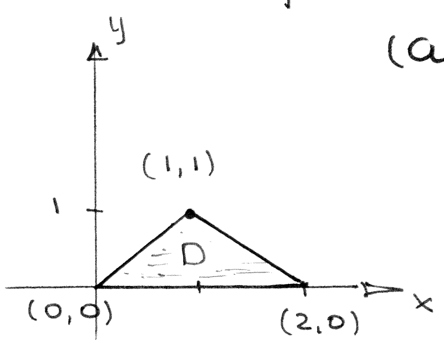
(b) Change the order of integration and evaluate it again.

$$\iint_D x\sqrt{y}\,dA = \int_0^1 \sqrt{y}\,dy \int_{y^2}^1 x\,dx =$$

$$= \int_0^1 \sqrt{y}\,dy\ \frac{x^2}{2}\Big]_{y^2}^1 = \frac{1}{2}\int_0^1 \sqrt{y}\,(1-y^4)\,dy = \frac{1}{2}\int_0^1 \left(y^{1/2} - y^{9/2}\right)dy$$

$$= \frac{1}{2}\left(\frac{2}{3}y^{3/2}\Big]_0^1 - \frac{2}{11}y^{11/2}\Big]_0^1\right) = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{11}\right) = \boxed{\frac{8}{33}}$$

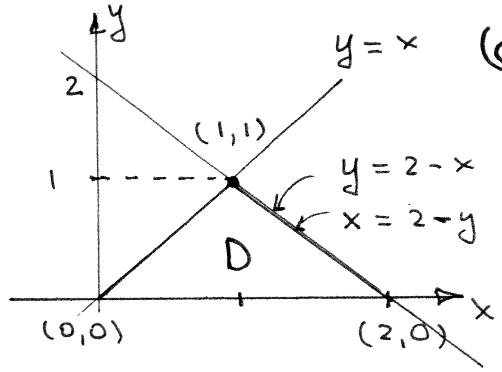
8. Given a solid in $\mathbb{R}^3$ bounded above by the parabolic cylinder $z = x^2$ and below by the region D in the xy-plane which is a triangle with vertices (0,0), (1,1), (2,0)



(a) Set up a double integral to evaluate the volume V of the solid

$$V = \iint_D x^2\,dA.$$

(b) Graph the region and set up a single iterated integral to find the volume (choose the appropriate order of integration



(c) $V = \iint_D x^2 dA =$

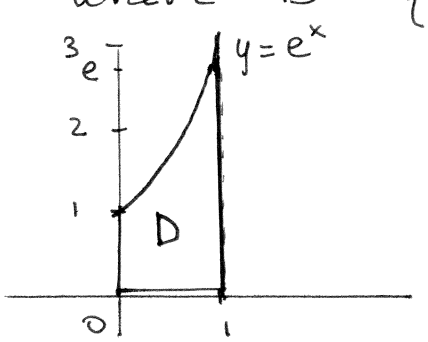
$$= \int_0^1 dy \int_y^{2-y} x^2 dx = \int_0^1 dy \left. \frac{x^3}{3} \right]_y^{2-y} =$$

$$= \frac{1}{3} \int_0^1 \left[(2-y)^3 - y^3 \right] dy =$$

$$= \frac{1}{3} \left(- \left. \frac{(2-y)^4}{4} \right]_0^1 - \left. \frac{y^4}{4} \right]_0^1 \right) = \frac{1}{12} \left[(16 - 1) - 1 \right] =$$

$$= \frac{1}{12} \cdot 14 = \boxed{\frac{7}{6}}$$

9. Evaluate a double integral $\iint_D \cos(e^x) dA$, where $D = \{ (x,y) \mid 0 \le x \le 1, 0 \le y \le e^x \}$.



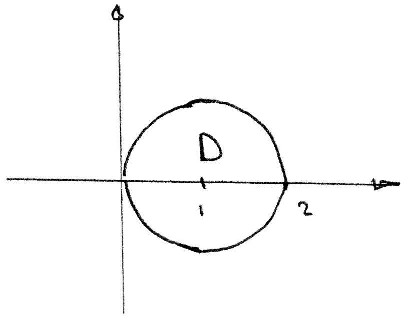
$$\iint_D \cos(e^x) dA = \int_0^1 \cos(e^x) dx \int_0^{e^x} dy =$$

$$= \int_0^1 e^x \cos(e^x) dx = \sin(e^x) \Big]_0^1 =$$

$$= \sin(e) - \sin 1.$$

10. Use polar coordinates to rewrite the double integral as iterated integral in two possible orders and reduce one of them into a single integral.

$\iint_D f(\sqrt{x^2+y^2}) dx dy$, where the region $D = \{ (x,y) \mid x^2+y^2 \le 2x \}$.



$$D = \{ (x,y) \mid x^2+y^2 \le 2x \}$$

$$x^2+y^2 \le 2x$$

$$(x^2-2x+1)+y^2 \le 1$$

$$(x-1)^2+y^2 \le 1$$

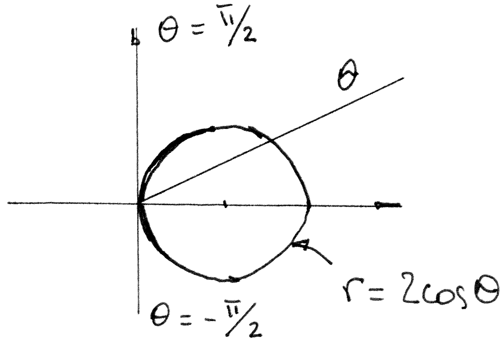
The boundary of D:

$$x^2+y^2 = 2x$$

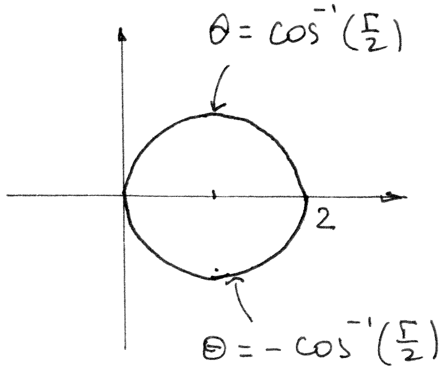
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta.$$



$$\iint_D f(\sqrt{x^2+y^2})\,dx\,dy = \boxed{\int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} r f(r)\,dr}$$



$$r = 2\cos\theta$$

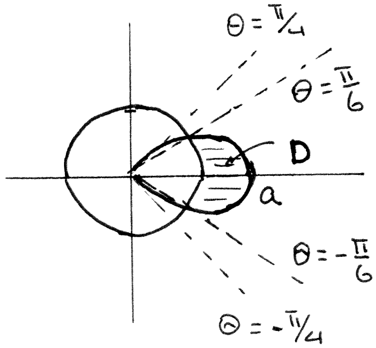
$$\cos\theta = \frac{r}{2}$$

$$\theta = \pm\cos^{-1}\left(\frac{r}{2}\right)$$

$$\iint_D f(\sqrt{x^2+y^2})\,dx\,dy = \boxed{\int_0^2 r f(r)\,dr \int_{-\cos^{-1}(r/2)}^{\cos^{-1}(r/2)} d\theta} =$$

$$= \int_0^2 r f(r)\cdot \theta \Big]_{-\cos^{-1}(r/2)}^{\cos^{-1}(r/2)} dr = \boxed{2\int_0^2 r\cos^{-1}(r/2) f(r)\,dr}$$

11. (a) Use polar coordinates to describe the region for $x \geq 0$ which is bounded by the lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$, $a>0$, and the circle $x^2+y^2=\frac{a^2}{2}$ (the part that lies outside the circle).



The lemniscate: $(x^2+y^2)^2 = a^2(x^2-y^2)$, $a>0$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$$

$$r^2 = a^2\cos 2\theta$$

$$r = a\sqrt{\cos 2\theta}$$

$$\cos 2\theta \geq 0$$

$$-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$$

$$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

The circle: $x^2+y^2=\frac{a^2}{2}$

$$r^2 = \frac{a^2}{2} \Rightarrow r = \frac{a}{\sqrt{2}}$$

Find θ at the points of intersection of the lemniscate and the circle:

$$\begin{cases} r^2 = a^2\cos 2\theta \\ r^2 = \frac{a^2}{2} \end{cases} \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \pm\cos^{-1}\left(\tfrac{1}{2}\right)$$

$$2\theta = \pm\frac{\pi}{3}$$

$$\theta = \pm\frac{\pi}{6}.$$

$$\boxed{D = \left\{ (r,\theta) \,\middle|\, -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6},\ \frac{a}{\sqrt{2}} \leq r \leq a\sqrt{\cos 2\theta} \right\}}$$

(b) Find the area of the region.

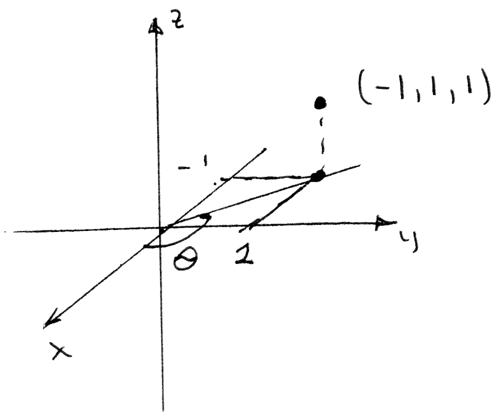
$$\text{Area} = \iint_D 1\,dx\,dy = \int_{-\pi/6}^{\pi/6} d\theta \int_{a/\sqrt{2}}^{a\sqrt{\cos 2\theta}} r\,dr = 2\int_0^{\pi/6} d\theta\, \frac{r^2}{2}\Bigg|_{\frac{a}{\sqrt{2}}}^{a\sqrt{\cos 2\theta}} =$$

$$= \int_0^{\pi/6}\left(a^2\cos 2\theta - \frac{a^2}{2}\right)d\theta = a^2\int_0^{\pi/6}\left(\cos 2\theta - \frac{1}{2}\right)d\theta =$$

$$= a^2\left(\frac{1}{2}\sin 2\theta\Big|_0^{\pi/6} - \frac{1}{2}\theta\Big|_0^{\pi/6}\right) = \frac{a^2}{2}\left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right) =$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \boxed{\frac{a^2}{12}\left(3\sqrt{3} - \pi\right)}$$

12. Convert the point $(-1,1,1)$ from cartesian to cylindrical and spherical coordinates, (r, θ, z) and (ρ, φ, θ), respectively.



In cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} x = -1 \\ y = 1 \\ z = 1 \end{cases}$$

$r = \sqrt{x^2+y^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = -1$ and θ is in Quadrant

$\Rightarrow \theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.

$z = z = 1$

$$\Rightarrow \boxed{(r, \theta, z) = \left(\sqrt{2}, \frac{3\pi}{4}, 1\right)}$$

In spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$\rho = \sqrt{x^2+y^2+z^2}$

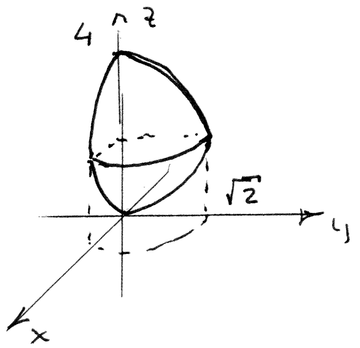
$\tan\theta = \frac{y}{x} = -1$ and θ is in Quadrant II

$\Rightarrow \theta = \frac{3\pi}{4}$

$\cos\varphi = \frac{z}{\rho} = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

$$\Rightarrow \boxed{(\rho, \varphi, \theta) = \left(\sqrt{3}, \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \frac{3\pi}{4}\right)}$$

13. (a) Describe in Cartesian coordinates the solid S that lies in the first octant and bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$.

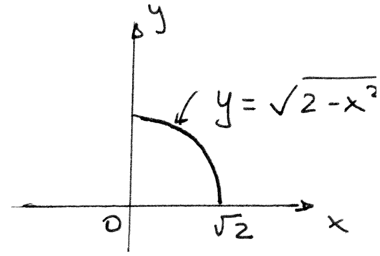


$$\begin{cases} z = x^2 + y^2 \\ z = 4 - x^2 - y^2 \end{cases}$$

$$x^2 + y^2 = 4 - x^2 - y^2$$

$$2(x^2 + y^2) = 4$$

$$x^2 + y^2 = 2$$



$$S = \{ (x, y, z) \mid 0 \le x \le \sqrt{2},\ 0 \le y \le \sqrt{2 - x^2},\ x^2 + y^2 \le z \le 4 - x^2 - y^2 \}.$$

(b) Find the volume V.

Describe the solid S in cylindrical coordinates.

$$S = \{ (r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{2},\ r^2 \le z \le 4 - r^2 \}$$

Volume V:

$$V = \iiint_S 1\, dx\, dy\, dz = \int_0^{\pi/2} d\theta \cdot \int_0^{\sqrt{2}} r\, dr \int_{r^2}^{4-r^2} dz =$$

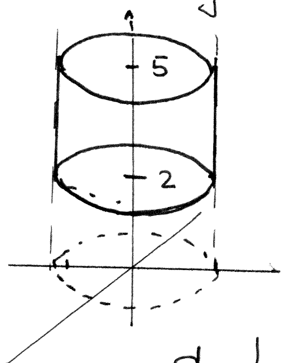
$$= \frac{\pi}{2} \int_0^{\sqrt{2}} r(4 - r^2 - r^2)\, dr = \frac{\pi}{2} \cdot \int_0^{\sqrt{2}} r(4 - 2r^2)\, dr$$

$$= \frac{\pi}{2} \cdot 2 \int_0^{\sqrt{2}} (2r - r^3)\, dr = \pi \left(r^2 - \frac{r^4}{4} \right) \Big|_0^{\sqrt{2}} =$$

$$= \pi \left((\sqrt{2})^2 - \frac{(\sqrt{2})^4}{4} \right) = \pi (2 - 1) = \boxed{\pi}$$

14. Describe the solids given in rectangular coordinates using the indicated coordinate systems. Set up iterated integrals of the function $f(x,y,z) = xy^2z$ over the solids. Tell which of them must be 0.

(a) The solid is a cylinder bounded by $x^2+y^2=4$ and planes $z=2$ and $z=5$ (in cylindrical coordinates).



Cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad |J| = r$$

$$S = \{(\theta, r, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ 2 \le z \le 5\}$$

$$f(x,y,z) = xy^2z = r^3\cos\theta\sin^2\theta\, z$$

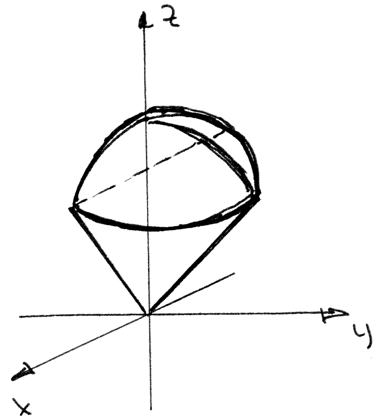
$$\iiint_S f(x,y,z)\, dV =$$

$$= \int_0^{2\pi} \cos\theta\sin^2\theta\, d\theta \cdot \int_0^2 r^4 dr \cdot \int_2^5 z\,dz = 0$$

Since

$$\int_0^{2\pi} \cos\theta\sin^2\theta\, d\theta = \frac{1}{3}\sin^3\theta\Big]_0^{2\pi} = 0$$

(b) The solid in octants I and II ($y \ge 0, z \ge 0$) is a half of a cone bounded by $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 9$ (in spherical coordinates)



$z = \sqrt{x^2+y^2}$

$x^2+y^2+z^2=9$

Spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad |J| = \rho^2\sin\varphi.$$

$x^2+y^2+z^2=9 \iff \rho^2 = 9 \iff \rho = 3$

$z = \sqrt{x^2+y^2} \iff \rho\cos\varphi = \rho\sin\varphi$

$\tan\varphi = 1$

$\varphi = \frac{\pi}{4}$

$S = \{(\theta,\varphi,\rho) \mid 0 \le \theta \le \pi,\ 0 \le \varphi \le \frac{\pi}{4},\ 0 \le \rho \le 3\}.$

$f(x,y,z) = xy^2z = \rho^4\cos\theta\sin^2\theta\sin^3\varphi\cos\varphi$

$\iiint\limits_S f(x,y,z)\,dV =$

$= \int_0^{\pi}\cos\theta\sin^2\theta\,d\theta \cdot \int_0^{\pi/4}\sin^4\varphi\cos\varphi\,d\varphi \cdot \int_0^3 \rho^6\,d\rho = 0$

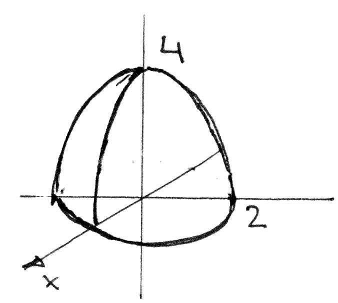
Since $\int_0^{\pi}\cos\theta\sin^2\theta\,d\theta = 0.$

(C) The solid in octants I and IV $(x \ge 0, z \ge 0)$ bounded by the plane $z=0$ and the paraboloid $z = 4-x^2-y^2$ (in cylindrical coordinates).

$z = 4-x^2-y^2 \Rightarrow z = 4-r^2$

$z = 0 \Rightarrow r = 2$

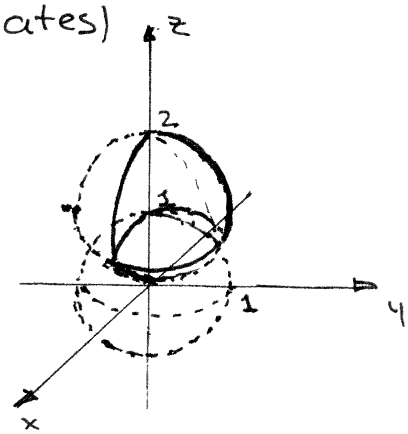
$f(x,y,z) = xy^2z = r^3\cos\theta\sin^2\theta\, z$



$$S = \{(\theta, r, z) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2,\ 0 \le z \le 4-r^2\}$$

$$\iiint_S f(x,y,z)\,dV = \int_{-\pi/2}^{\pi/2} \cos\theta \sin^2\theta\, d\theta \int_0^2 r^4 dr \int_0^{4-r^2} z\,dz$$

(d) The solid in octants I and II $(y \ge 0, z \ge 0)$ bounded below by the sphere $x^2+y^2+z^2=1$ and bounded above by the sphere $x^2+y^2+z^2=2z$ (in spherical coordinates)



$$x^2+y^2+z^2=1 \Rightarrow \rho^2 = 1$$
$$\rho = 1$$

$$x^2+y^2+z^2=2z \Rightarrow \rho^2 = 2\rho\cos\varphi$$
$$\rho = 2\cos\varphi$$

$$y \ge 0 \Rightarrow 0 \le \theta \le \pi$$

$\rho = 1$ and $\rho = 2\cos\varphi$

$$2\cos\varphi = 1$$
$$\cos\varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$$
$$0 \le \varphi \le \frac{\pi}{3}$$
$$1 \le \rho \le 2\cos\varphi$$

$$S = \{(\theta, \varphi, \rho) \mid 0 \le \theta \le \pi,\ 0 \le \varphi \le \frac{\pi}{3},\ 1 \le \rho \le 2\cos\varphi\}$$

$$f(x,y,z) = xy^2z = \rho^4 \cos\theta \sin^2\theta \sin^3\varphi \cos\varphi$$

$$|J| = \rho^2 \sin\varphi$$

$$\iiint_S f(x,y,z)\,dV = \int_0^{\pi} \cos\theta \sin^2\theta\, d\theta \cdot \int_0^{\pi/3} \sin^4\varphi \cos\varphi\, d\varphi \int_1^{2\cos\varphi} \rho^6 d\rho$$

$$\iiint_S f(x,y,z)\,dV = 0 \quad \text{since} \int_0^{\pi} \cos\theta \sin^2\theta\, d\theta = 0.$$

15. Use an appropriate coordinate system to integrate the function $f(x,y,z) = x^2 z$ over the solid bounded by the surfaces $z = \sqrt{x^2+y^2}$ and $z = 2 - \sqrt{x^2+y^2}$.

In cylindrical coordinates:

$z = \sqrt{x^2+y^2} \Rightarrow z = r$

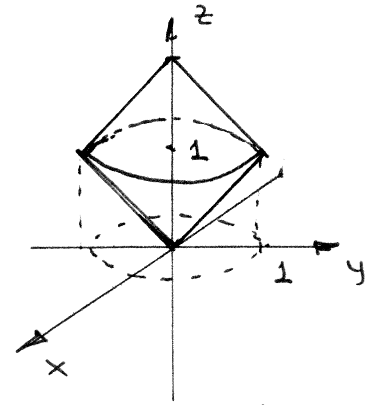
$z = 2 - \sqrt{x^2+y^2} \Rightarrow z = 2 - r$

$z = r$ and $z = 2 - r$

$r = 2 - r$

$2r = 2$

$r = 1$



$S = \{(\theta, r, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le 2 - r\}$

$|J| = r$

$f(x,y,z) = x^2 z = r^2 \cos^2\theta\, z$

$$\iiint_S f(x,y,z)\,dV = \int_0^{2\pi} \cos^2\theta\, d\theta \cdot \int_0^1 r^3 dr \int_r^{2-r} z\, dz =$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos 2\theta)\, d\theta \cdot \int_0^1 r^3 dr\ \frac{z^2}{2}\Big]_r^{2-r} =$$

$$= \frac{1}{4} \cdot 2\pi \int_0^1 r^3 \left[(2-r)^2 - r^2\right] dr =$$

$$= \frac{\pi}{2} \int_0^1 r^3 (4 - 4r + \cancel{r^2} - \cancel{r^2})\, dr =$$

$$= 4 \cdot \frac{\pi}{2} \int_0^1 r^3 (1-r)\, dr = 2\pi \int_0^1 (r^3 - r^4)\, dr$$

$$= 2\pi \left(\frac{r^4}{4} - \frac{r^5}{5}\right)\Big]_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{5}\right) = 2\pi \frac{1}{20}$$

$$= \boxed{\frac{\pi}{10}}$$

16. Use spherical coordinates to evaluate the integral $\iiint\limits_E z\,dV$, where E is the solid in the first octant bounded by the surfaces $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$, $z^2=x^2+y^2$, $z^2=\frac{1}{3}(x^2+y^2)$ and $y=x$, $y=2x$.

1) The solid is located between two vertical planes $y=x$ and $y=2x$:

$$y=x \iff \rho\sin\theta\sin\varphi=\rho\cos\theta\sin\varphi$$
$$\tan\theta=1 \qquad (0<\theta<\tfrac{\pi}{2})$$
$$\theta=\frac{\pi}{4}$$

$$y=2x \iff \rho\sin\theta\sin\varphi=2\rho\cos\theta\sin\varphi$$
$$\tan\theta=2 \qquad (0<\theta<\tfrac{\pi}{2})$$
$$\theta=\tan^{-1}2$$

$$\Rightarrow \quad \frac{\pi}{4}\le\theta\le\tan^{-1}2$$

2) The solid is located between two cones:

$$z^2=x^2+y^2 \iff \rho^2\cos^2\varphi=\rho^2\sin^2\varphi$$
$$\tan^2\varphi=1 \qquad (0<\varphi<\tfrac{\pi}{2})$$
$$\tan\varphi=1$$
$$\varphi=\frac{\pi}{4}$$

$$z^2=\frac{1}{3}(x^2+y^2) \iff \rho^2\cos^2\varphi=\frac{1}{3}\rho^2\sin^2\varphi$$
$$\tan^2\varphi=3 \qquad (0<\varphi<\tfrac{\pi}{2})$$
$$\tan\varphi=\sqrt{3}$$
$$\varphi=\frac{\pi}{3}$$

$$\Rightarrow \quad \frac{\pi}{4}\le\varphi\le\frac{\pi}{3}$$

3) The solid is located between two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$

$x^2+y^2+z^2=1 \iff \rho=1$

$x^2+y^2+z^2=4 \iff \rho=2$

$\Rightarrow 1 \le \rho \le 2$

4) $f(x,y,z) = z = \rho\cos\varphi$

$|J| = \rho^2 \sin\varphi$

$$\iiint_E z\,dV = \int_{\pi/4}^{\tan^{-1}2} d\theta \cdot \int_{\pi/4}^{\pi/3} \sin\varphi\cos\varphi\, d\varphi \cdot \int_1^2 \rho^3\, d\rho =$$

$$= \left(\tan^{-1}2 - \frac{\pi}{4}\right) \left.\frac{\sin^2\varphi}{2}\right]_{\pi/4}^{\pi/3} \left.\frac{\rho^4}{4}\right]_1^2 =$$

$$= \frac{1}{8}\left(\tan^{-1}2 - \frac{\pi}{4}\right)\left(\sin^2\frac{\pi}{3} - \sin^2\frac{\pi}{4}\right)\cdot(16-1)$$

$$= \frac{15}{8}\left(\tan^{-1}2 - \frac{\pi}{4}\right)\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right) =$$

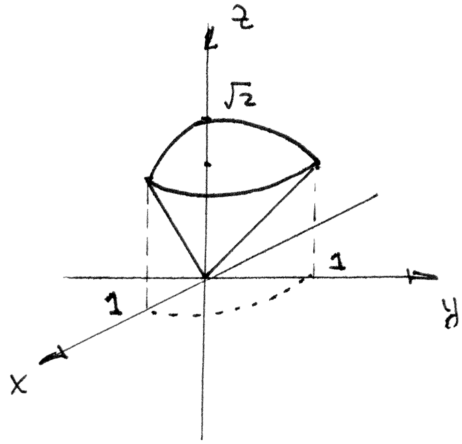
$$= \frac{15}{8}\cdot\frac{1}{4}\left(\tan^{-1}2 - \frac{\pi}{4}\right)(3-2)$$

$$= \boxed{\frac{15}{32}\left(\tan^{-1}2 - \frac{\pi}{4}\right)}$$

17. Evaluate the integral by changing it to spherical coordinates.

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)\, dz$$

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$



$$|J| = \rho^2\sin\varphi$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$z = \sqrt{x^2+y^2} \iff \varphi = \frac{\pi}{4}$$

$$0 \le \varphi \le \frac{\pi}{4}$$

$$z = \sqrt{2-x^2-y^2}$$

$$x^2+y^2+z^2 = 2, \quad z \geq 0$$

$$\rho^2 = 2 \implies \rho = \sqrt{2}$$

$$0 \le \rho \le \sqrt{2}$$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)\,dz =$$

$$= \int_0^{\pi/2} d\theta \cdot \int_0^{\pi/4} \sin\varphi\, d\varphi \cdot \int_0^{\sqrt{2}} \rho^4 d\rho = \frac{\pi}{2}(-\cos\varphi)\Big]_0^{\pi/4} \frac{\rho^5}{5}\Big|_0^{\sqrt{2}} =$$

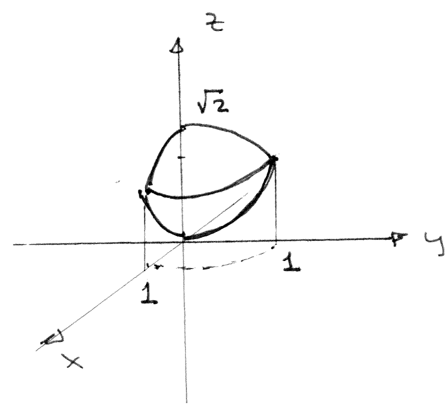
$$= \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{5}(\sqrt{2})^5 = \frac{\pi}{2}\,\frac{\sqrt{2}-1}{\sqrt{2}}\cdot\frac{1}{5}\cdot 4\sqrt{2} =$$

$$= \boxed{\frac{2\pi}{5}(\sqrt{2}-1)}$$

18. Evaluate the integral by changing it to cylindrical coordinates:

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz.$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad |J| = r$$



$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le r \le 1$$

$$z = x^2 + y^2 \iff z = r^2$$

$$z = \sqrt{2 - x^2 - y^2} \iff z = \sqrt{2 - r^2}$$

$$r^2 \le z \le \sqrt{2 - r^2}$$

$$S = \left\{ (\theta, r, z) \,\middle|\, 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1,\ r^2 \le z \le \sqrt{2 - r^2} \right\}.$$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz = \int_0^{\pi/2} d\theta \int_0^1 r\,dr \int_{r^2}^{\sqrt{2-r^2}} z\,dz =$$

$$= \frac{\pi}{2} \int_0^1 r \left. \frac{z^2}{2} \right|_{r^2}^{\sqrt{2-r^2}} dr = \frac{\pi}{4} \int_0^1 r\,(2 - r^2 - r^4)\,dr =$$

$$= \frac{\pi}{4} \int_0^1 (2r - r^3 - r^5)\,dr = \frac{\pi}{4} \left(r^2 - \frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^1 =$$

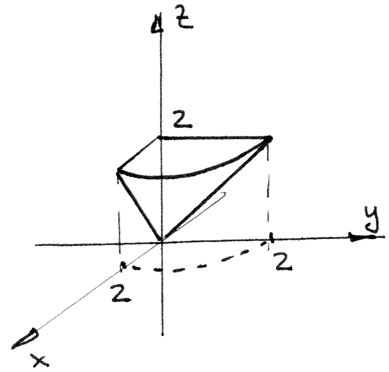
$$= \frac{\pi}{4} \left(1 - \frac{1}{4} - \frac{1}{6} \right) = \frac{\pi}{4} \; \frac{12 - 3 - 2}{12} = \boxed{\frac{7\pi}{48}}$$

19. Sketch the solid whose volume is given by the iterated integral in spherical coordinates:

$$V = \int_0^{\pi/2} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^{\frac{2}{\cos\varphi}} \rho^2 \, d\rho.$$

$$S = \{ (\theta, \varphi, \rho) \mid 0 \le \theta \le \frac{\pi}{2}, \ 0 \le \varphi \le \frac{\pi}{4}, \ 0 \le \rho \le \frac{2}{\cos\varphi} \}.$$

(a) Describe the solid in Cartesian coordinates.



$$\varphi = \frac{\pi}{4} \iff z = \sqrt{x^2+y^2}$$

$$\rho = \frac{2}{\cos\varphi} \iff \rho\cos\varphi = 2$$

$$z = 2$$

$$\sqrt{x^2+y^2} \le z \le 2$$

$z = \sqrt{x^2+y^2}$ and $z = 2$

$$\Rightarrow \quad x^2 + y^2 = 4 \qquad (0 \le x \le 2, \ y \ge 0)$$

$$\Rightarrow \quad y = \sqrt{4-x^2}, \quad 0 \le x \le 2$$

$$S = \{ (x, y, z) \mid 0 \le x \le 2, \ 0 \le y \le \sqrt{4-x^2}, \ \sqrt{x^2+y^2} \le z \le 2 \}$$

(b) Rewrite the integral in cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad |J| = r$$

$$z = \sqrt{x^2+y^2} \Rightarrow z = r$$

$$S = \{ (\theta, r, z) \mid 0 \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2, \ r \le z \le 2 \}$$

$$V = \int_0^{\pi/2} d\theta \int_0^2 r\,dr \int_r^2 dz$$

(c) Calculate the volume V. Verify your answer by using the formula from Geometry.

$$V = \int_0^{\pi/2} d\theta \int_0^2 r\,dr \int_r^2 dz = \frac{\pi}{2}\int_0^2 r \; z\Big]_r^2 dr = \frac{\pi}{2}\int_0^2 r(2-r)\,dr$$

$$= \frac{\pi}{2}\int_0^2 (2r - r^2)\,dr = \frac{\pi}{2}\left(r^2 - \frac{r^3}{3}\right)\Big]_0^2 = \frac{\pi}{2}\left(4 - \frac{8}{3}\right)$$

$$= \frac{\pi}{2} \; \frac{12-8}{3} = \frac{\pi}{2}\cdot\frac{4}{3} = \boxed{\frac{2\pi}{3}}$$

By the Formula for the volume of a cone:

$$V = \frac{1}{3}\,[\text{Area of Base}] \times [\text{Height}]$$

$$V = \frac{1}{3}\left(\frac{1}{4}\pi(2)^2\right)\cdot 2 = \boxed{\frac{2\pi}{3}}$$

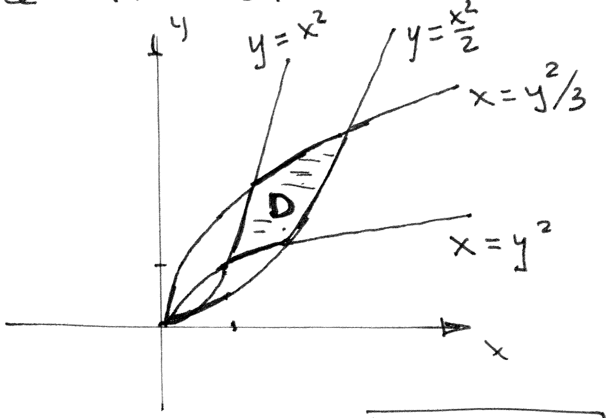
20. The region D in $\mathbb{R}^2$ is bounded by the curves $x^2 = y$, $x^2 = 2y$, $y^2 = x$, $y^2 = 3x$.

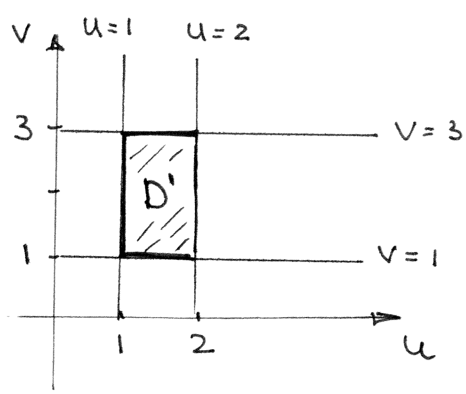
(a) Find the coordinate transformation T^{-1}: $T^{-1}(x,y) = (u,v)$ that maps the region D in the xy-plane onto a rectangle in the uv-plane.

$$T^{-1}: \quad u = \frac{x^2}{y}, \quad v = \frac{y^2}{x} \qquad (x, y \in D)$$

$$T^{-1}: \quad D \to D', \quad \text{where } D' = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le 3\}$$

(b) Graph the region and its image under the transformation defined in (a).





(c) Without finding the transformation T itself, evaluate its Jacobian, $J(u,v)$.

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2x}{y} \cdot \frac{2y}{x} - \frac{x^2}{y^2} \cdot \frac{y^2}{x^2} =$$

$$= 4 - 1 = 3$$

$$\Rightarrow \quad J(u,v) = \frac{1}{3}.$$

(d) Find the area of the region D using the new coordinate system.

$$\text{Area}(D) = \iint_D 1\,dx\,dy = \iint_{D'} |J|\,du\,dv = \frac{1}{3}\iint_{D'} 1\,du\,dv$$

$$= \frac{1}{3}\,\text{Area}(D') = \frac{1}{3}\cdot 1 \cdot 2 = \boxed{\frac{2}{3}}$$

21. Compute the Jacobian of the transformation: $x = u$, $y = \frac{v}{u}$, $z = w$.

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} =$$

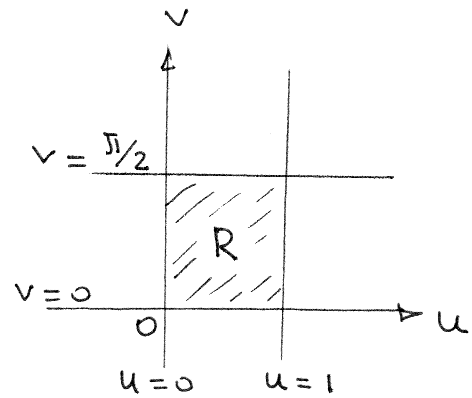
$$= \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \boxed{\frac{1}{u}}$$

22. (a) Describe the image (in Cartesian coordinates) of the region

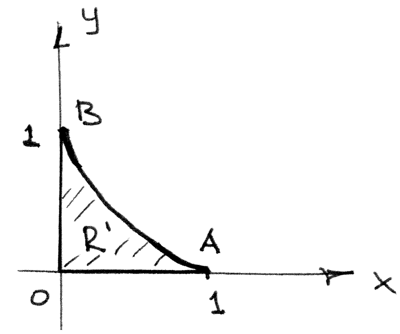
$R = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le \frac{\pi}{2}\}$

under the transformation

$$T(u, v) = (u \cos^3 v,\ u \sin^3 v).$$



$$\begin{cases} x = u \cos^3 v \\ y = u \sin^3 v \end{cases}$$



$$\begin{cases} u = 0 \\ 0 \le v \le \frac{\pi}{2} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

– the origin $(0,0)$

$$\begin{cases} u = 1 \\ 0 \le v \le \frac{\pi}{2} \end{cases} \Rightarrow \begin{cases} x = \cos^3 v \\ y = \sin^3 v \end{cases} \quad \left(0 \le v \le \frac{\pi}{2}\right)$$

$$x^{2/3} + y^{2/3} = 1 \quad (x, y \geq 0)$$

– the arc AB.

$$\begin{cases} v = 0 \\ 0 \le u \le 1 \end{cases} \Rightarrow \begin{cases} x = u \\ y = 0 \end{cases} \quad (0 \le u \le 1)$$

– the line segment OA.

$$\begin{cases} v = \frac{\pi}{2} \\ 0 \le u \le 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = u \end{cases} \quad (0 \le u \le 1)$$

– the line segment OB.

The region R' is the image of the region R under the transformation T.

(b) Find the Jacobian, $J(u, v)$, of the transformation defined above.

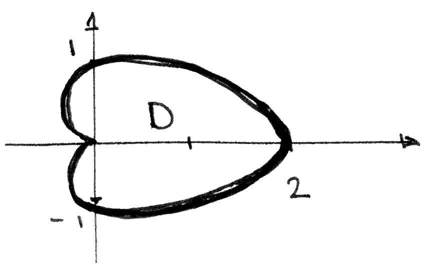
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} =$$

$$= \begin{vmatrix} \cos^3 v & 3u\cos^2 v\,(-\sin v) \\ \sin^3 v & 3u \sin^2 v \cos v \end{vmatrix} =$$

$$= 3u\,(\cos^4 v \sin^2 v + \cos^2 v \sin^4 v)$$

$$= \boxed{3u \cos^2 v \sin^2 v}$$

23. Find the area of the region D bounded by the cardioid $r = 1 + \cos\theta$, $0 \le \theta \le 2\pi$. Graph the region.



$$\text{Area}(D) = \iint_D 1\,dA =$$

$$= 2\int_0^{\pi} d\theta \int_0^{1+\cos\theta} r\,dr =$$

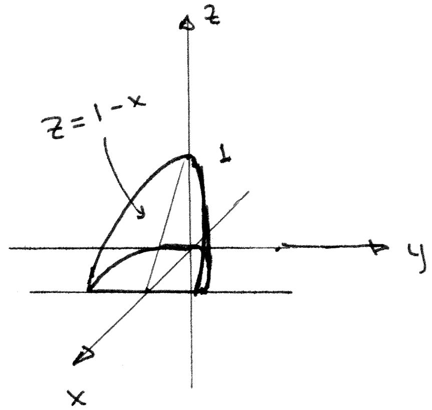
$$= 2\int_0^{\pi} d\theta \left.\frac{r^2}{2}\right|_0^{1+\cos\theta} =$$

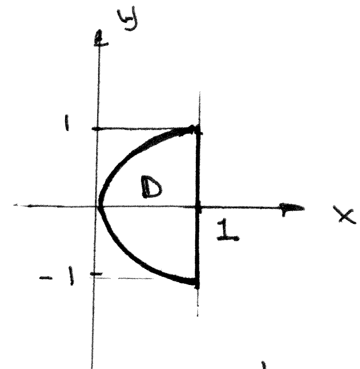
$$= \int_0^{\pi} (1+\cos\theta)^2\,d\theta = \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \theta\Big|_0^{\pi} + 2\sin\theta\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} (1+\cos 2\theta)\,d\theta =$$

$$= \pi + 0 + \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$$

24. (a) Find the volume of the solid S bounded by the parabolic cylinder $x = y^2$ and by the planes $z = 0$ and $x + z - 1 = 0$. Graph the solid.





$$D = \{(x,y) \mid -1 \le y \le 1,\ y^2 \le x \le 1\}$$

$$\text{Volume} = V = \iint_D (1-x)\,dA = 2\int_0^1 dy \int_{y^2}^{1} (1-x)\,dx$$

$$= 2\int_0^1 dy \left(-\frac{(1-x)^2}{2}\right)\Bigg]_{y^2}^{1} = \int_0^1 (1-y^2)^2\,dy =$$

$$= \int_0^1 (1-2y^2+y^4)\,dy = \left(y - \frac{2}{3}y^3 + \frac{y^5}{5}\right)\Bigg]_0^1 =$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{15-10+3}{15} = \boxed{\frac{8}{15}}$$

(b) Find the average value of the function $f(x,y,z) = x$ over the solid S described in (a).

$$\text{Average value} = \frac{1}{V}\iiint_S x\,dV.$$

$$S = \{(x,y,z) \mid -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le 1-x\}$$

$$V = \frac{8}{15}$$

$$\iiint_S x\,dV = \int_{-1}^{1} dy \int_{y^2}^{1} x\,dx \int_0^{1-x} dz = 2\int_0^1 dy \int_{y^2}^{1} x\,dx \int_0^{1-x} dz$$

$$= 2\int_0^1 dy \int_{y^2}^{1} (x-x^2)\,dx = 2\int_0^1 dy \left(\frac{x^2}{2}\Big|_{y^2}^{1} - \frac{x^3}{3}\Big|_{y^2}^{1} \right)$$

$$= 2\int_0^1 \left(\frac{1}{2}(1-y^4) - \frac{1}{3}(1-y^6) \right) dy =$$

$$= 2\left(\frac{1}{2}\left(y - \frac{y^5}{5}\right)\Big]_0^1 - \frac{1}{3}\left(y - \frac{y^7}{7}\right)\Big|_0^1 \right) =$$

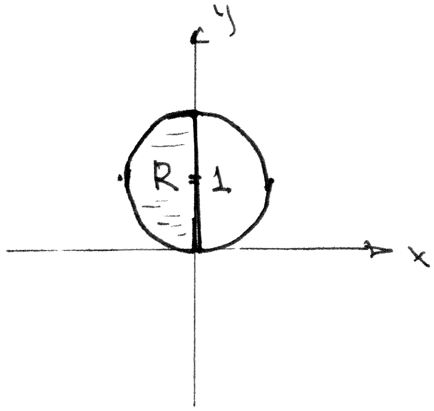
$$= 2\left(\frac{1}{2}\left(1-\frac{1}{5}\right) - \frac{1}{3}\left(1-\frac{1}{7}\right) \right) = 2\left(\frac{1}{2}\cdot\frac{4}{5} - \frac{1}{3}\cdot\frac{6}{7} \right)$$

$$= 2\left(\frac{2}{5} - \frac{2}{7} \right) = 4\,\frac{7-5}{35} = \frac{8}{35}$$

Average Value $= \frac{1}{V}\iiint_S x\,dV = \frac{15}{8}\cdot\frac{8}{35} = \boxed{\frac{3}{7}}$

25. Find the image under the transformation $x = r\cos\theta$, $y = r\sin\theta$ of the polar region $0 \le r \le 2\sin\theta$, $\frac{\pi}{2} \le \theta \le \pi$.

$R = \{(\theta, r) \mid \frac{\pi}{2} \le \theta \le \pi,\ 0 \le r \le 2\sin\theta\}$.



$r = 2\sin\theta \qquad (\frac{\pi}{2} \le \theta \le \pi)$

$r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1$

$$R: \begin{cases} x^2 + (y-1)^2 \le 1 \\ x \le 0 \end{cases}$$