Review 3: L17 – L22

3. Consider the function $f(x,y) = x + y^2$ on the rectangle $R = \{(x,y) \mid 0 \le x \le 2, -1 \le y \le 1\}$. Evaluate $\iint_R f(x,y)\,dA$.
4. Find the volume of a solid beneath the plane $f(x,y) = 3x + 3y + 9$ and above the region $R = \{(x,y) \mid -1 \le x \le 1, 1 \le y \le 3\}$.
5. Find the volume of the tetrahedron bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 2y + z = 2$. Do it in three ways: (1) using a formula from Geometry; (2) using a double integral; (3) using a triple integral.
6. Given an iterated integral: $\int_0^1 \int_0^3 4xye^{x^2}\,dxdy = 2\int_0^1 ydy \int_0^3 2xe^{x^2}\,dx$
   (a) Rewrite it as a product of two single integrals if possible;
   (b) Evaluate the integral.
7. Given an iterated integral: $\int_0^1 \int_0^{\sqrt{x}} x\sqrt{y}\ dydx = \int_0^1 xdx \int_0^{\sqrt{x}} \sqrt{y}\ dy$
   (a) Evaluate the integral;
   (b) Change the order of integration and evaluate it again. (Make sure your answers match!)
8. Given a solid in $\mathbb{R}^3$ bounded above by the parabolic cylinder $z = x^2$ and below by a region D in the xy-plane which is a triangle with vertices $(0,0),(1,1),(2,0)$.
   (a) Set up a double integral to evaluate the volume V of the solid;
   (b) Graph the region and set up a single iterated integral to find the volume (choose the appropriate order of integration);
   (c) Evaluate the integral.
9. Evaluate a double integral $\iint_D \cos(e^x)\,dA$, where $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le e^x\}$.
10. Use polar coordinates to rewrite the double integral as iterated integrals in two possible orders and reduce one of them into a single integral.
    $\iint_D f\left(\sqrt{x^2 + y^2}\right)dA$, where the region $D = \{(x,y) \mid x^2 + y^2 \le 2x\}$.

11. (a) Use polar coordinates to describe the region for $x \geq 0$ which is bounded by the lemniscate $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, $a>0$, and the circle $x^2+y^2=\dfrac{a^2}{2}$ (the part that lies outside the circle).
(b) Find the area of the region.

12. Convert the point $(-1,1,1)$ from Cartesian to cylindrical and spherical coordinates, (r,θ,z) and (ρ,ϕ,θ), respectively.

13. (a) Describe in Cartesian coordinates the solid S that lies in the first octant and bounded by the paraboloids $z=x^2+y^2$ and $z=4-x^2-y^2$.
(b) Find its volume V. (You will need to change the coordinate system).

14. Describe the solids given in rectangular coordinates using the indicated coordinate systems. Set up iterated integrals of the function $f(x,y,z)=xy^2z$ over the solids. Tell which of them must be 0.

(a) The solid is a cylinder bounded by $x^2+y^2=4$ and planes $z=2$ and $z=5$ (in cylindrical coordinates);

(b) The solid in octants I and II $(y\geq 0,\ z\geq 0)$ is half of a cone bounded by $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=9$ (in spherical coordinates);

(c) The solid in octants I and IV $(x\geq 0,\ z\geq 0)$ bounded by the plane $z=0$ and the paraboloid $z=4-x^2-y^2$ (in cylindrical coordinates);

(d) The solid in octants I and II $(y\geq 0,\ z\geq 0)$ bounded below by the sphere $x^2+y^2+z^2=1$ and bounded above by the sphere $x^2+y^2+z^2=2z$ (in spherical coordinates).

15. Use an appropriate coordinate system to integrate the function $f(x,y,z)=x^2z$ over the solid bounded by the surfaces $z=\sqrt{x^2+y^2}$ and $z=2-\sqrt{x^2+y^2}$.

16. Use spherical coordinates to evaluate the integral $\iiint_E z\,dV$, where E is the solid in the first octant bounded by the surfaces $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$, $z^2=x^2+y^2$, $z^2=\frac{1}{3}\left(x^2+y^2\right)$ and $y=x,\ y=2x$.

17. Evaluate the integral by changing it to spherical coordinates.

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \left(x^2+y^2+z^2\right) dz .$$

18. Evaluate the integral by changing it to cylindrical coordinates: $\displaystyle\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz$

19. Sketch the solid whose volume is given by the iterated integral in spherical coordinates.

$$\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} \sin\phi\, d\phi \int_0^{\frac{2}{\cos\varphi}} \rho^2\, d\rho .$$

(a) Describe the solid in Cartesian coordinates.
(b) Rewrite the integral in cylindrical coordinates.

(c) Calculate the volume V. Verify your answer by using the formula from Geometry.

20. The region D in $\mathbb{R}^2$ is bounded by the curves $x^2 = y,\ x^2 = 2y,\ y^2 = x,$ and $y^2 = 3x$.

(a) Find the coordinate transformation $T^{-1} : T^{-1}(x, y) = (u, v)$ that maps the region D in the $xy-$plane onto a rectangle in the $uv-$plane.

(b) Graph the region and its image under the transformation defined in (a).

(c) Without finding the transformation T itself, evaluate its Jacobian $J(u, v)$;

(d) Find the area of the region D using the new coordinate system.

21. Compute the Jacobian for the transformation: $x = u,\ y = \dfrac{v}{u},\ z = w$.

22. (a) Describe the image (in Cartesian coordinates) of the region $R = \left\{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le \dfrac{\pi}{2}\right\}$ under the transformation $T(u, v) = \left(u\cos^3 v,\ u\sin^3 v\right)$.

(b) Find the Jacobian, $J(u, v)$, of the transformation defined above.

23. Find the area of the region D bounded by the cardioid $r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi$. Graph the region.

24. (a) Find the volume of the solid S bounded by the parabolic cylinder $x = y^2$ and by the planes $z = 0$ and $x + z - 1 = 0$. Graph the solid.

(b) Find the average value of the function $f(x, y, z) = x$ over the solid S described in (a).

25. Find the image under the transformation $x = r\cos\theta,\ y = r\sin\theta$ of the polar region

$0 \le r \le 2\sin\theta,\ \dfrac{\pi}{2} \le \theta \le \pi$.