Calculus I: MAC2311
Spring 2024
Exam 2 A
2/28/2024
Time Limit: 100 Minutes

Name: Key

Section: ______________

UF-ID: ______________

Scantron Instructions: This exam uses a scantron. Follow the instructions listed on this page to fill out the scantron.

A. Sign your scantron **on the back** at the bottom in the white area.

B. Write **and code** in the spaces indicated:

1) Name (last name, first initial, middle initial)

2) UFID Number

3) 4-digit Section Number

C. Under *special codes*, code in the test numbers 2, 1:

1 • 3 4 5 6 7 8 9 0
• 2 3 4 5 6 7 8 9 0

D. At the top right of your scantron, fill in the *Test Form Code* as A .

• B C D E

E. This exam consists of 14 multiple choice questions and 5 free response questions. Make sure you check for errors in the number of questions your exam contains.

F. The time allowed is 100 minutes.

G. WHEN YOU ARE FINISHED:

1) Before turning in your test check for **transcribing errors**. Any mistakes you leave in are there to stay!

2) You must turn in your scantron and free response packet to your proctor. **Be prepared to show your proctor a valid GatorOne ID or other signed ID.**

It is your responsibility to ensure that your test has **19 questions**. If it does not, show it to your proctor immediately. You will not be permitted to make up any problems omitted from your test after the testing period ends. There are a total of 105 points available on this exam.

Part I Instructions: 14 multiple choice questions. Complete the scantron sheet provided with your information and fill in the appropriate spaces to answer your questions. Only the answer on the scantron sheet will be graded. Each problem is worth five (5) points for a total of 70 points on Part I.

quotient rule.

1. Find the equation of the tangent line to $f(x) = \frac{x+2}{3x}$ at $x = -1$.

(A) $y = -\frac{2}{3}x + 1$ (B) $y = -\frac{2}{3}x - 1$

(C) $y = \frac{2}{3}x + 1$ (D) $y = \frac{2}{3}x - 1$

$f(-1) = \frac{1}{-3}$

$x_1 \quad y_1$

$(-1, -\frac{1}{3})$

$m = -\frac{2}{3}$

Find slope:

$f'(x) = \frac{(3x)(1) - (x+2)(3)}{(3x)^2}$

$f'(-1) = \frac{-3 - (1)(3)}{(-3)^2} = \frac{-6}{9} = -\frac{2}{3}$ (slope)

$y - (-\frac{1}{3}) = -\frac{2}{3}(x - (-1))$

$y + \frac{1}{3} = -\frac{2}{3}(x + 1)$

$y = -\frac{2}{3}x - \frac{2}{3} - \frac{1}{3}$

$\boxed{y = -\frac{2}{3}x - 1}$

2. Let $f(x) = e^{2\sin(x)}$. Compute $f''(0)$. (That is, find the second derivative at $x = 0$)

(A) 0 (B) 2

(C) -2 (D) 4 (E) -4

① ②

$f'(x) = e^{2\sin(x)} \cdot 2\cos(x)$

$f''(x) = e^{2\sin(x)}(-2\sin(x)) + e^{2\sin(x)} \cdot 2\cos(x)(2\cos(x))$ ← product rule

$f''(0) = e^0(-2 \cdot 0) + e^0 \cdot 2(1)2 \cdot (1)$ plug in $x=0$ immediately.

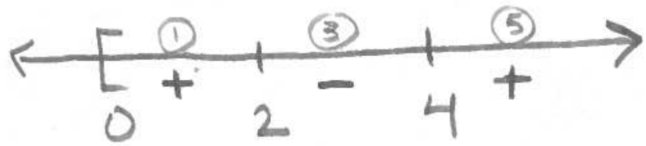
$= 0 + 4 = \boxed{4}$

3. Let $s(t) = t^3 - 9t^2 + 24t$ be the position function of a particle given in feet after t seconds. Assume $t \geq 0$. On which interval does the particle have **both** positive velocity and positive acceleration?

(A) $(3, 4)$ (B) $(0, 2)$ (C) $(3, \infty)$ (D) $(4, \infty)$ (E) $(2, 4)$

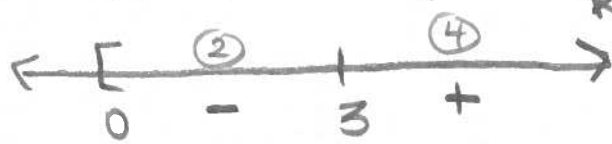
$v(t) = s'(t) = 3t^2 - 18t + 24$

$= 3(t^2 - 6t + 8) = 3(t-2)(t-4)$



Test numbers here

$a(t) = v'(t) = 6t - 18 = 6(t-3)$



Test numbers here

Both are positive on $(4, \infty)$.

4. Calculate the derivative of $y = e^{2x} + \arcsin(e^x)$.

(A) $y' = 2e^{2x} + \frac{1}{\sqrt{1-e^{2x}}}$ (B) $y' = 2e^{2x} + \frac{1}{\sqrt{1+e^{2x}}}$ (C) $y' = 2e^{2x} - \frac{e^x}{\sqrt{1+e^{2x}}}$

(D) $y' = e^{2x} + \frac{e^x}{\sqrt{1-e^x}}$ (E) $y' = 2e^{2x} + \frac{e^x}{\sqrt{1-e^{2x}}}$

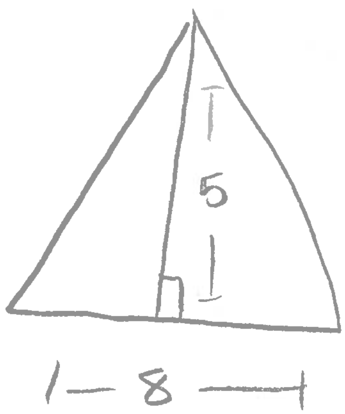
Recall: $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$

$$y' = 2e^{2x} + \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

$$= 2e^{2x} + \frac{e^x}{\sqrt{1-e^{2x}}}$$

5. The area of a triangle is increasing at a rate of 4 cm^2 per minute while the height is increasing at a rate of 2 cm per minute. At the moment the area is 20 cm^2 and the height is 5 cm, how fast is the base changing?
Recall: $A_{triangle} = \frac{1}{2}bh$.

(A) $-\frac{16}{5}$ cm per minute (B) $-\frac{8}{5}$ cm per minute (C) $-\frac{4}{5}$ cm per minute (D) $-\frac{2}{5}$ cm per minute



$A = \frac{1}{2}bh$

currently,

$20 = \frac{1}{2}b(5)$. So,

$b = 8$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + b\frac{dh}{dt}\right)$

$4 = \frac{1}{2}\left(\frac{db}{dt} \cdot 5 + 8 \cdot 2\right)$

$8 = 5\frac{db}{dt} + 16 \longrightarrow \boxed{-\frac{8}{5} = \frac{db}{dt}}$

$\frac{dA}{dt} = 4\text{cm}^2/\text{min}$

$\frac{dh}{dt} = 2\text{cm}/\text{min}$

$h = 5$

$A = 20$

$b = 8$

$\frac{db}{dt} = ?$

6. Find the derivative of $f(x) = \ln(x^2 + 2x)$ at $x = 1$.

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{4}{3}$

(D) $\frac{1}{3}$ (E) $\frac{1}{2}$

$f'(x) = \frac{1}{x^2+2x}(2x+2)$

$f'(1) = \frac{1}{3}(4) = 4/3$

7. Use properties of logarithms to find the slope of the tangent line to $y = \ln\left(\frac{e^{3x}}{\sqrt{2x+1}(x+1)^5}\right)$ at $x = 0$.

(A) 7 (B) $-\frac{5}{2}$ (C) -8

(D) -3 (E) -9

$y = \ln(e^{3x}) - \ln((2x+1)^{1/2} \cdot (x+1)^5)$

$y = 3x\ln(e) - [\ln(2x+1)^{1/2} + \ln(x+1)^5]$

$y = 3x - \frac{1}{2}\ln(2x+1) - 5\ln(x+1)$

$y' = 3 - \frac{1}{\cancel{2}(2x+1)} \cdot \cancel{2} - \frac{5}{x+1}$ (slope)

$y'(0) = 3 - \frac{1}{1} - \frac{5}{1} = \boxed{-3}$

8. For which of the following functions is it **NOT** necessary to use logarithmic differentiation to calculate its derivative?

(A) $f(x) = x^{\ln(x)}$ (B) $g(x) = e^{e^x}$ (C) $h(x) = (2x+1)^{\cos(x)}$ (D) $k(x) = x^x$

You don't need log. diff. for $g(x) = e^{e^x}$.
Its derivative can be calculated using chain rule:
$g'(x) = e^{e^x} \cdot e^x$

product rule

9. Let $f(x) = \cos(2x) + x\ln(x)$. Calculate $f'(1)$.

(A) $1 - 2\sin(2)$ (B) $1 + 2\sin(2)$ (C) $1 - 2\cos(2)$ (D) $1 + 2\cos(2)$ (E) 2

$$f'(x) = -2\sin(2x) + (1)\ln(x) + x\left(\frac{1}{x}\right)$$

$$f'(1) = -2\sin(2) + \cancel{\ln(1)}^{0} + 1$$

$$\boxed{f'(1) = 1 - 2\sin(2)}$$

10. Suppose $K(x) = f(g(x)) + x^2 - h(x)$. You are given that $K'(1) = 10$, $g(1) = 2$, $g'(1) = 4$, $f'(2) = -2$, and $f'(4) = -4$. Find $h'(1)$.

(A) -18 (B) -16

(C) -14 (D) -12

$K'(x) = f'(g(x)) \cdot g'(x) + 2x - h'(x)$

$K'(1) = f'(g(1)) \cdot g'(1) + 2(1) - h'(1)$ (plug in $x=1$)

$10 = f'(2) \cdot (4) + 2 - h'(1)$ (use given info)

$10 = (-2)(4) + 2 - h'(1)$

$10 = -6 - h'(1)$

$16 = -h'(1)$

$\boxed{h'(1) = -16}$

11. How many of the following statements are **true**?

(i) $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$. ✓

~~(ii)~~ $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$. Wrong. $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

~~(iii)~~ $\frac{d}{dx}\left(2^x\right) = 2^x$. Wrong. $\frac{d}{dx}(2^x) = 2^x \ln(2)$

(iv) $\frac{d}{dx}\left(2^\pi\right) = 0$. ✓

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

12. Find the derivative using implicit differentiation:

$$\sin(xy) = x^2 y$$

(A) $y' = \frac{y\cos(xy)-2xy}{x^2+x\cos(xy)}$ (B) $y' = \frac{y\cos(xy)+2y}{x^2-x\cos(xy)}$

(C) $y' = \frac{y\cos(xy)-2xy}{x^2-\cos(xy)}$ (D) $y' = \frac{y\cos(xy)-2xy}{x^2-x\cos(xy)}$

$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(x^2y)$ solve for y'.

$\cos(xy)[y+xy'] = 2xy + x^2y'$

$y\cos(xy) + xy'\cos(xy) = 2xy + x^2y'$

$xy'\cos(xy) - x^2y' = 2xy - y\cos(xy)$

$y'(x\cos(xy) - x^2) = 2xy - y\cos(xy)$

$$\boxed{y' = \frac{2xy - y\cos(xy)}{x\cos(xy) - x^2} = \frac{y\cos(xy) - 2xy}{x^2 - x\cos(xy)}}$$

13. Suppose that $x^2 + y^3 = 28$. Find $\frac{dx}{dt}$ when $x = 1$ and $\frac{dy}{dt} = 2$.

(A) $\frac{dx}{dt} = -27$ (B) $\frac{dx}{dt} = -20$ (C) $\frac{dx}{dt} = -17$

(D) $\frac{dx}{dt} = -10$ (E) None of these.

Take derivative with respect to t:

$$\frac{d}{dt}(x^2+y^3) = \frac{d}{dt}(28)$$

Note: when $x=1$, $1^2+y^3=28$ $\therefore y=3$

$$2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

plug in $x=1$, $y=3$, $\frac{dy}{dt}=2$.

$$2(1)\frac{dx}{dt} + 3(3)^2(2) = 0$$

$$2\cdot\frac{dx}{dt} + 27(2) = 0$$

$$\boxed{\frac{dx}{dt} = -27}$$

14. Calculate the 2024th derivative of $f(x) = 2e^{-2x}$. (Hint: Find the first few derivatives and look for a pattern)

(A) $f^{(2024)}(x) = 2^{2024}e^{-2x}$ (B) $f^{(2024)}(x) = -2^{2024}e^{-2x}$

(C) $f^{(2024)}(x) = 2^{2025}e^{-2x}$ (D) $f^{(2024)}(x) = -2^{2025}e^{-2x}$

$$f(x) = 2e^{-2x}$$

$$f'(x) = -2(2)e^{-2x} = (-1)^1 2^2 e^{-2x}$$

$$f''(x) = (-2)^2(2)e^{-2x} = (-1)^2 2^3 e^{-2x}$$

$$f'''(x) = (-2)^3(2)e^{-2x} = (-1)^3 2^4 e^{-2x}$$

$\vdots$

* The power of (-1) is the same as the derivative number.

* The power of 2 is one more than the derivative number.

continuing this pattern...

$$f^{(2024)}(x) = (-1)^{2024} 2^{2025} e^{-2x} = \boxed{2^{2025}e^{-2x}}$$

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Part II Instructions: 5 free response questions. Neatly give a complete solution to each problem and show all work and intermediate steps. We are grading the work and notation as well as the answer. Each problem is worth seven (7) points. A total of 35 points is possible on Part II. **No credit will given without proper work.** If we cannot read it and follow it, you will receive no credit for the problem.

For Instructor Use Only:

FR 1	
FR 2	
FR 3	
FR 4	
FR 5	
Total Points	

1. Calculate the derivative of the following functions. You do **NOT** need to simplify your final answer for any of these.

(i) $f(x) = (x^4 - 3x^{-5})^4$

$f'(x) = 4(x^4 - 3x^{-5})^3 (4x^3 + 15x^{-6})$

(ii) $g(x) = e^{-x}(x^2 - 2x + 2)$

$g'(x) = (e^{-x})(2x-2) + (-e^{-x})(x^2 - 2x + 2)$

(iii) $h(x) = x\tan^{-1}(2x)$

$h'(x) = (1)\tan^{-1}(2x) + \frac{x}{1+(2x)^2} \cdot 2$

$h'(x) = \tan^{-1}(2x) + \frac{2x}{1+4x^2}$

(iv) $k(x) = \cos\left(e^{\sqrt{\sin(x)}}\right) = \cos\left(e^{(\sin x)^{1/2}}\right)$

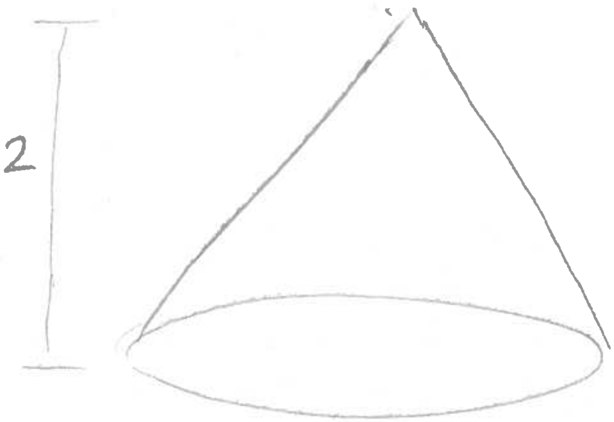
$k'(x) = -\sin\left(e^{(\sin(x))^{1/2}}\right) \cdot \left(e^{(\sin x)^{1/2}}\right)'$

$k'(x) = -\sin\left(e^{(\sin(x))^{1/2}}\right) \cdot e^{(\sin x)^{1/2}} \cdot \left[(\sin x)^{1/2}\right]'$

$k'(x) = -\sin\left(e^{(\sin(x))^{1/2}}\right) \cdot e^{(\sin x)^{1/2}} \cdot \frac{1}{2}(\sin x)^{-1/2} \cdot \cos x$

2. Sand pouring from a chute forms a conical pile whose height is always equal to half of the diameter. If the height of the pile increases at a constant rate of 6 feet per minute, at what rate is the sand pouring from the chute when the pile is 12 feet high? The volume formula for a cone is $V_{cone} = \frac{\pi}{3}r^2h$.

$h = \frac{1}{2}d = \frac{1}{2}(2r) = r$.



$\frac{dh}{dt} = 6$ ft/min

$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}h^3$ since $h=r$

$\frac{dV}{dt} = \frac{\pi}{\cancel{3}}(\cancel{3}h^2)\frac{dh}{dt}$

$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$ plug in given

$\frac{dV}{dt} = \pi \cdot 12^2 \cdot 6$

$\frac{dV}{dt} = 144 \cdot 6\pi = \boxed{864\pi \text{ ft}^3/\text{min}}$

3. The position of a particular particle is given by the equation

$$s(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and $s(t)$ in meters. Suppose $t \geq 0$. Answer the following:

(i) Find the velocity at time t.

$v(t) = s'(t) = 3t^2 - 12t + 9$

(ii) When is the particle at rest?

Need $v(t) = 0$

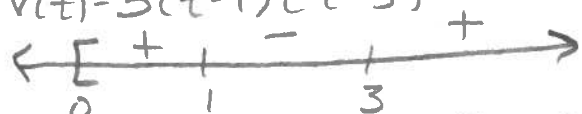
$\therefore 0 = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$\boxed{t = 1}$ and $\boxed{t = 3}$

(iii) On which intervals of time is the particle **slowing down**? Use interval notation for your final answer.

We need velocity and acceleration to have opposite signs:

$v(t) = 3(t-1)(t-3)$



$a(t) = 6t - 12 = 6(t-2)$

Slowing down: $\boxed{(0,1) \cup (2,3)}$

(iv) Find the total distance traveled by the particle during the first four seconds.

Split up calculation based on when the particle turns around.

Total distance $= |s(1) - s(0)| + |s(3) - s(1)| + |s(4) - s(3)|$

$= |4 - 0| + |0 - 4| + |4 - 0|$

$= 4 + 4 + 4 = \boxed{12 \text{ meters}}$

4. Assume $x > 0$. Use **logarithmic differentiation** to find the derivative of

$$y = (\cos(x))^{\ln(x)}$$

Write your final answer in terms of x.

$$\ln(y) = \ln\left[(\cos(x))^{\ln(x)}\right]$$

$$\ln(y) = \ln(x) \cdot \ln(\cos(x)) \quad \text{(product rule)}$$

$$\frac{1}{y} \cdot y' = \left(\frac{1}{x}\right)\ln(\cos(x)) + \ln(x)\left(\frac{1}{\cos x} \cdot (-\sin x)\right)$$

$$y' = \left[\frac{\ln(\cos(x))}{x} - (\tan x)\ln(x)\right] y$$

$$y' = \left(\frac{\ln(\cos(x))}{x} - \tan(x) \cdot \ln(x)\right)[\cos(x)]^{\ln(x)}$$

5. Suppose

$$\sin(x+y) = y^2 \cos(x).$$

(i) Use implicit differentiation to find $y' = \frac{dy}{dx}$.

$$\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}(y^2\cos(x))$$

$$\cos(x+y)\cdot[1+y'] = 2yy'\cos(x) - y^2\sin(x)$$

$$\cos(x+y) + y'\cos(x+y) = 2yy'\cos(x) - y^2\sin(x)$$

$$y'\cos(x+y) - 2yy'\cos(x) = -y^2\sin(x) - \cos(x+y)$$

$$\boxed{y' = \frac{-y^2\sin(x) - \cos(x+y)}{\cos(x+y) - 2y\cos(x)}}$$

x_1 y_1

(ii) Find the equation of the tangent line to the equation in part (i) at the point $(\frac{\pi}{2}, -\frac{\pi}{2})$.

$$y'\left(\frac{\pi}{2}, \frac{-\pi}{2}\right) = \frac{-\left(\frac{-\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}-\frac{\pi}{2}\right) - 2\left(\frac{-\pi}{2}\right)\cancel{\cos}\left(\frac{\pi}{2}\right)}$$

0

$$\text{Slope} = \frac{-\frac{\pi^2}{4}(1) - \cos(0)}{\cos(0) + 0} = \boxed{-\frac{\pi^2}{4} - 1} = m$$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y + \frac{\pi}{2} = \left(-\frac{\pi^2}{4} - 1\right)\left(x - \frac{\pi}{2}\right)}$$