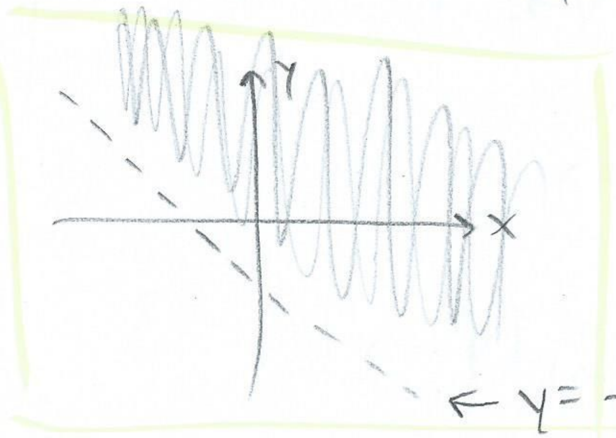
1) Find and sketch the domain of each function.

① $f(x,y) = \ln(x+y+1)$

Natural log needs a positive input, so $x+y+1>0$. This becomes $y>-x-1$

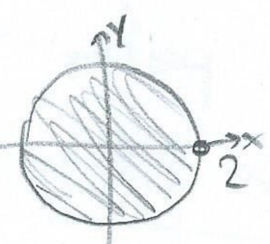



② $f(x,y) = \sqrt{4-x^2-y^2} + \sqrt{1-x^2}$

$4-x^2-y^2 \geq 0$

$4 \geq x^2+y^2$

$\Downarrow$

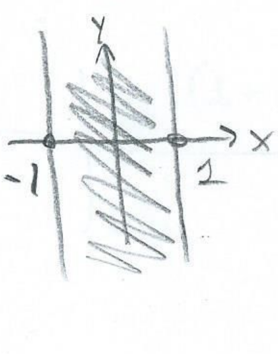



$1-x^2 \geq 0$

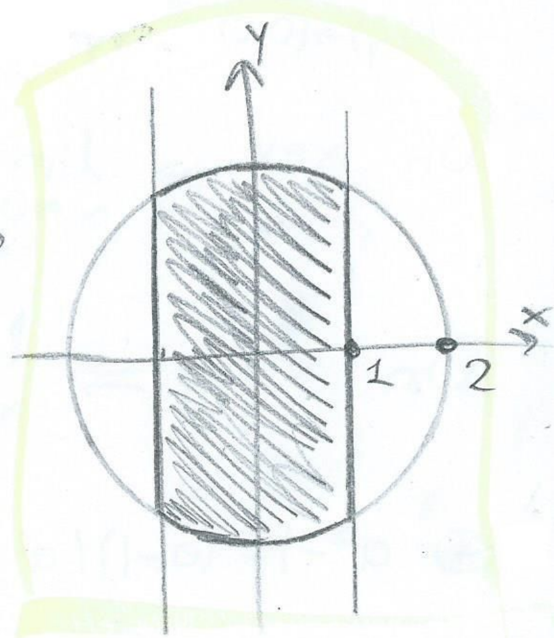
$1 \geq x^2$

$-1 \leq x \leq 1$

$\Downarrow$




$\Rightarrow$



2) Show that the limit does not exist.

① $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy^2-1}{y-1}$

On the path $x=1$

$$\lim_{y\to 1} \frac{y^2-1}{y-1} = \lim_{y\to 1} (y+1) = 2$$

On the path $y=x$

$$\lim_{y\to 1} \frac{y^3-1}{y-1} \overset{*}{=} \lim_{y\to 1} (y^2+y+1) = 3$$

Two paths, two limits.

② $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{xy^2}$

On $x=y \Rightarrow \lim\limits_{x\to 0} \dfrac{x^3+x^3}{x^3} = \lim\limits_{x\to 0} \dfrac{2x^3}{x^3} = 2$

On $x=-y \Rightarrow \lim\limits_{x\to 0} \dfrac{-y^3+y^3}{-y^3} = 0$

$*$ $a^3-1 = (a-1)(a^2+a+1)$

Two paths, two limits

3) Evaluate the following limits

① $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = \lim_{(x,y)\to(1,1)} \frac{(xy)^3-1}{xy-1} = \lim_{(x,y)\to(1,1)} \left[(xy)^2+xy+1\right] = 3$

② $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{(2,2)} \frac{\sqrt{x+y}^{\,2}-2^2}{\sqrt{x+y}-2} = \lim_{(2,2)} \sqrt{x+y}+2 = 4$

③ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y\sin(2x)}{x} = \left(\lim_{y\to0} e^y\right)\left(\lim_{x\to0} 2\cdot\frac{\sin(2x)}{2x}\right)$

$= 1\cdot 2 = 2$

FACT

$\displaystyle\lim_{u\to0}\frac{\sin(u)}{u} = 1$

④ $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{u\to0} u\ln(u) \text{ "=" } 0\cdot-\infty$

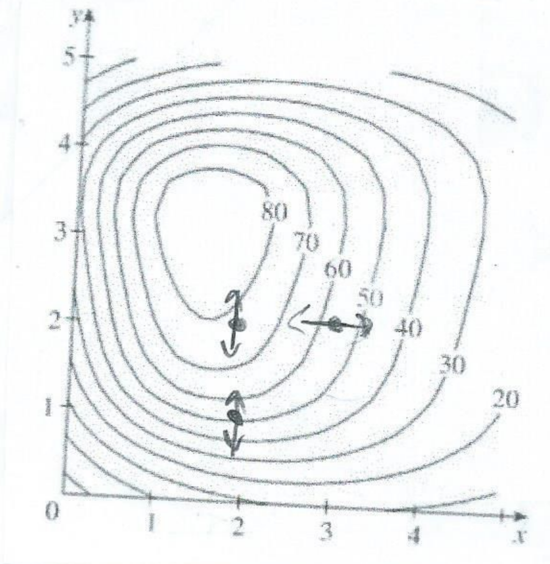
$u = x^2+y^2$

$(x,y)\to(0,0) \Rightarrow u\to0$

$\displaystyle= \lim_{u\to0}\frac{\ln(u)}{1/u} \overset{L'H}{=} \lim_{u\to0}\frac{1/u}{-1/u^2}$

$\displaystyle= \lim_{u\to0}(-u) = 0$

4) The contour map of a function $f$ is shown.




① Is $f_x(3,2)$ positive or negative?

It's negative. At $(3,2)$, if you move a little to the right (positive $\Delta x$), the value of $f$ goes from 50 to 40 (negative $\Delta f$).

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{\text{neg.}}{\text{pos.}} = \text{neg.}$$

② Which is greater, $f_y(2,1)$ or $f_y(2,2)$?

$f_y(2,1)$ is greater. Near $(2,1)$ the contour lines are closer together, so for a fixed $\Delta f$, you need a smaller $\Delta y$. Since $\frac{df}{dy} \approx \frac{\Delta f}{\Delta y}$, a small denominator gives a big number.

5) Consider the function $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

① Is f continuous at (0,0)?

Path $x=0 \Rightarrow \lim_{y \to 0} \dfrac{\sin(0)}{0^2+y^2} = 0$

Path $x=y \Rightarrow \lim_{x \to 0} \dfrac{\sin(x^2)}{x^2+x^2} = \dfrac{1}{2} \lim_{x \to 0} \dfrac{\sin(x^2)}{x^2} = \dfrac{1}{2}$

f doesn't even have a limit at (0,0), so it's not continuous.

② Is f differentiable at (0,0)?

No. If it were, it would also have to be continuous at (0,0), but we already know that it isn't.

6)

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

① Is f continuous at (0,0)?

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0$$

$x = r\cos\theta$
$y = r\sin\theta$

It isn't continuous because $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$.

② Can you redefine f to be continuous at (0,0)?

Yes, just say $f(0,0)=0$.

---

7) Compute all first and second derivatives of $f(x,y) = x^y$

$f_x = yx^{y-1}$

$f_y = \ln(x)\,x^y$

$f_{xx} = y(y-1)x^{y-2}$

$f_{yy} = (\ln(x))^2 x^y$

$f_{xy} = f_{yx} = x^{y-1} + y\ln(x)\,x^{y-1}$

$\frac{d}{dx}[a^x]$

$= \ln(a)\cdot a^x$

8) Find the linear approximation of $f(x,y,z) = x^3\sqrt{y^2+z^2}$ at $(2,3,4)$ and use it to estimate $f(1.98, 3.02, 4.01)$.

The linear approximation at this point will be given by

$$L(x,y,z) = f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) + f(2,3,4)$$

$$f_x = 3x^2\sqrt{y^2+z^2} \implies f_x(2,3,4) = 12\sqrt{9+16} = 12\cdot 5 = 60$$

$$f_y = \frac{x^3 y}{\sqrt{y^2+z^2}} \implies f_y(2,3,4) = \frac{2^3\cdot 3}{\sqrt{9+16}} = \frac{24}{5}$$

$$\frac{d}{dx}\left[\sqrt{x^2+a^2}\right] = \frac{x}{\sqrt{x^2+a^2}}$$

$$f_z = \frac{x^3 z}{\sqrt{y^2+z^2}} \implies f_z(2,3,4) = \frac{2^3\cdot 4}{5} = \frac{32}{5}$$

$$f(2,3,4) = 2^3\sqrt{9+16} = 8\cdot 5 = 40$$

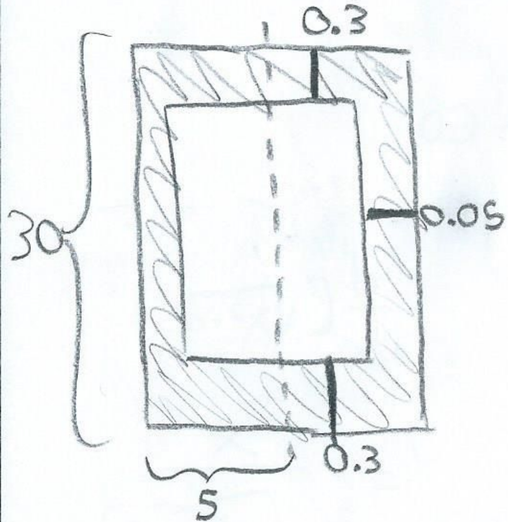
$$L(x,y,z) = 60(x-2) + \frac{24}{5}(y-3) + \frac{32}{5}(z-4) + 40$$

$$f(1.98, 3.02, 4.01) \simeq L(1.98, 3.02, 4.01)$$

$$= 60(-0.02) + \frac{24}{5}(0.02) + \frac{32}{5}(0.01) + 40 = 38.96$$

⊛ In reality, $f(1.98, 3.02, 4.01) = 38.96728\ldots$
Pretty good estimate.

9) Use differentials to estimate the amount of metal in a closed cylindrical can that is 30 cm high and 5 cm in radius if the metal in the top and the bottom is 0.3 cm thick and the metal in the sides is 0.05 cm thick.




$$V = \pi r^2 h$$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$\Delta V \simeq \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial h} \Delta h$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}[\pi r^2 h] = 2\pi r h \quad \& \quad \frac{\partial V}{\partial h} = \frac{\partial}{\partial h}[\pi r^2 h] = \pi r^2$$

For this can, $\frac{\partial V}{\partial r} = 2\pi(5)(30) = 300\pi$ & $\frac{\partial V}{\partial h} = \pi(5)^2 = 25\pi$

So $\Delta V = (300\pi)(0.05) + (25\pi)(0.6)$

$= 15\pi + 15\pi = 30\pi \text{ cm}^3$

10) Find $\partial z/\partial x$ and $\partial z/\partial y$ at $(0,1,2)$ if $x-yz+\cos(xyz)=2$.

You could do implicit differentiation or the following:

Set $F(x,y,z)=x-yz+\cos(xyz)$. Then

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$$

$F_x = 1 - yz\sin(xyz)$ so $F_x(0,1,2) = 1-2\sin(0) = 1$

$F_y = -z - xz\sin(xyz)$ so $F_y(0,1,2) = -2-0 = -2$

$F_z = -y - xy\sin(xyz)$ so $F_z(0,1,2) = -1-0 = -1$

$$\frac{\partial z}{\partial x} = -\frac{1}{-1} = 1$$

$$\frac{\partial z}{\partial y} = -\frac{-2}{-1} = -2$$

11) Find an equation of the tangent plane to the surface $z = x\sin(x+y)$ at the point $(-1, 1, 0)$.

Rewrite as $\underbrace{x\sin(x+y) - z}_{F(x,y,z)} = 0$

The normal vector for the surface will be the gradient of $F$, $\vec{\nabla} F$.

$$F_x = \sin(x+y) + x\cos(x+y) \qquad F_x(-1,1,0) = \sin(-1+1) - \cos(1-1) = -1$$

$$F_y = x\cos(x+y) \quad \xRightarrow{\text{at } (-1,1,0)} \quad F_y(-1,1,0) = -1\cos(-1+1) = -1$$

$$F_z = -1 \qquad F_z(-1,1,0) = -1$$

$\vec{\nabla} F(-1,1,0) = \langle -1, -1, -1 \rangle$ so the plane is going to be given by $\langle -1, -1, -1 \rangle \cdot \langle x+1, y-1, z-0 \rangle = 0$

$$\Rightarrow -x - 1 - y + 1 - z = 0$$

$$\Rightarrow \boxed{x + y + z = 0}$$

12) Let $z=\sqrt{x^2+y^2}$. Show $\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2$

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}} \Longrightarrow \frac{\partial^2 z}{\partial x^2}=\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}} \Longrightarrow \frac{\partial^2 z}{\partial y^2}=\frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]=\frac{\partial}{\partial x}\left[\frac{y}{\sqrt{x^2+y^2}}\right]=\frac{-xy}{(x^2+y^2)^{3/2}}$$

So $$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}=\frac{x^2y^2}{(x^2+y^2)^3}$$

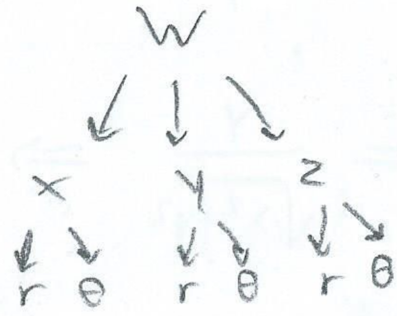
and $$\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=\left(\frac{-xy}{(x^2+y^2)^{3/2}}\right)^2=\frac{x^2y^2}{(x^2+y^2)^3}$$

13) Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ when $r=2$ and $\theta=\pi/2$ if

$$w = xy + yz + xz, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta.$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

and similar for $\frac{\partial w}{\partial \theta}$.




$$\frac{\partial w}{\partial x} = y+z \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial w}{\partial y} = x+z \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial w}{\partial z} = x+y \qquad \frac{\partial z}{\partial r} = \theta \qquad \frac{\partial z}{\partial \theta} = r$$

At $(r,\theta) = (2, \pi/2)$, we have $x=0$, $y=2$, $z=\pi$.

$$\frac{\partial w}{\partial r} = (2+\pi)(0) + (\pi)(1) + (2)(\pi/2) = 2\pi$$

$$\frac{\partial w}{\partial \theta} = (2+\pi)(-2) + (\pi)(0) + (2)(2) = -4 - 2\pi + 4 = -2\pi$$

14) Find the directional derivative of $f(x,y) = x^2 e^{-y}$ at the point $(-2,0)$ in the direction toward the point $(2,-3)$.

$D_{\hat{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \hat{u}$ where $\hat{u}$ is a <u>unit</u> vector in the desired direction.

$\langle -2,0\rangle + \vec{u} = \langle 2,-3\rangle \Rightarrow \vec{u} = \langle 4,-3\rangle$ but this isn't unit.

$|\vec{u}| = 5$ so $\hat{u} = \langle 4/5, -3/5\rangle$.

$\vec{\nabla} f = \langle f_x, f_y\rangle = \langle 2xe^{-y}, -x^2e^{-y}\rangle$

$\vec{\nabla} f(-2,0) = \langle -4,-4\rangle$ so

$D_{\hat{u}} f(-2,0) = \langle -4,-4\rangle \cdot \langle 4/5, -3/5\rangle = \frac{-16}{5} + \frac{12}{5} = \frac{-4}{5}$

15) Let $f(x,y) = \ln(1+xy)$.

① Find the unit vectors in the direction of steepest ascent and steepest descent at $(1,2)$.

Observe $D_{\hat{u}}f = \vec{\nabla}f \cdot \hat{u} = |\vec{\nabla}f||\hat{u}|\cos\theta = |\vec{\nabla}f|\cos\theta$.
So $D_{\hat{u}}f$ is as positive as possible if $\theta = 0$ and as negative as possible if $\theta = \pi$. So $\vec{\nabla}f$ gives the direction of steepest ascent.

$$\vec{\nabla}f = \left\langle \frac{y}{1+xy}, \frac{x}{1+xy} \right\rangle \Longrightarrow \vec{\nabla}f(1,2) = \left\langle \frac{2}{3}, \frac{1}{3} \right\rangle$$

This isn't unit, so just consider $\vec{u} = \langle 2, 1\rangle$.

$|\vec{u}| = \sqrt{5}$ so $\boxed{\begin{array}{l} \hat{u} = \langle 2/\sqrt{5}, 1/\sqrt{5}\rangle \text{ steepest ascent,} \\ -\hat{u} = \langle -2/\sqrt{5}, -1/\sqrt{5}\rangle \text{ steepest descent.} \end{array}}$

② Find a unit vector pointing in a direction of no change at $(1,2)$.

From before, $D_{\hat{u}}f = 0$ means $\theta = \pi/2$ or $\theta = 3\pi/2$, so our vector should be $90°$ to our gradient.

$$\hat{u} = \langle 2/\sqrt{5}, 1/\sqrt{5}\rangle \Longrightarrow \boxed{\hat{v} = \langle -1/\sqrt{5}, 2/\sqrt{5}\rangle}$$

$\hat{v}$ is $90°$ to $\hat{u}$ since $\hat{v} \cdot \hat{u} = 0$.

$\langle a,b\rangle \cdot \langle -b,a\rangle$
$= -ab + ba$
$= 0$

16) Find equations of ① the tangent plane and ② the normal line to the surface $xy+yz+xz=5$ at $(1,2,1)$.

① Similar to question 11, find normal vector by $\vec{\nabla} F$ where $F=xy+yz+xz$. $\vec{\nabla}F=\langle y+z,\ x+z,\ x+y\rangle$, so $\vec{F}(1,2,1)=\langle 3,2,3\rangle$

$$\langle 3,2,3\rangle\cdot\langle x-1,\ y-2,\ z-1\rangle=0$$

$$\Rightarrow 3x-3+2y-4+3z-3=0$$

$$\Rightarrow 3x+2y+3z=10$$

② Using the same vector $\langle 3,2,3\rangle$,

$$\vec{r}(t)=\langle 3,2,3\rangle t+\langle 1,2,1\rangle$$

$$=\langle 3t+1,\ 2t+2,\ 3t+1\rangle$$

Or by solving each equation for $t$,

$$\frac{x-1}{3}=\frac{y-2}{2}=\frac{z-1}{3}$$

$x=3t+1 \to x-1=3t \to \frac{x-1}{3}=t$

$y=2t+2 \to y-2=2t \to \frac{y-2}{2}=t$

$z=3t+1 \longrightarrow \frac{z-1}{3}=t$

17) Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersect the paraboloid a second time?

$$F(x,y,z) = x^2 + y^2 - z = 0$$

$$\vec{\nabla} F = \langle 2x, 2y, -1 \rangle \Rightarrow \vec{\nabla} F(1,1,2) = \langle 2,2,-1 \rangle$$

$$\vec{r}(t) = \langle 2,2,-1 \rangle t + \langle 1,1,2 \rangle = \langle \underbrace{2t+1}_{x}, \underbrace{2t+1}_{y}, \underbrace{-t+2}_{z} \rangle.$$

To find the intersection, plug the x, y, z of the line into the paraboloid's equation.

$$-t+2 = (2t+1)^2 + (2t+1)^2 = 8t^2 + 8t + 2$$

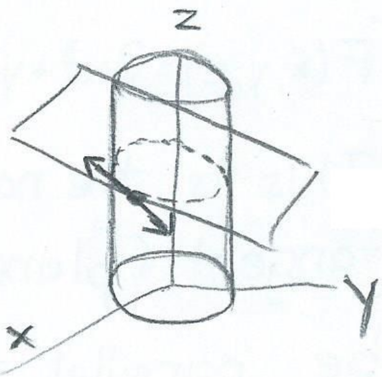
$$\Rightarrow 8t^2 + 9t = 0 \Rightarrow t(8t+9) = 0 \quad \begin{cases} t = 0 \\ t = -9/8 \end{cases}$$

$t=0$ is the point $(1,1,2)$. We need the other point.

$$\vec{r}(-9/8) = \left\langle 2\left(\tfrac{-9}{8}\right)+1,\ 2\left(\tfrac{-9}{8}\right)+1,\ \tfrac{9}{8}+2 \right\rangle$$

$$= \langle -5/4, -5/4, 25/8 \rangle$$

18) The plane $y+z=3$ intersects the cylinder $x^2+y^2=5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1,2,1)$.




The line we want is tangent to this curve, and the curve lies inside both the two surfaces. So it is tangent to each surface, and in particular, 90° to each surface's normal vector.

$$P(x) = y+z=3 \Rightarrow \vec{\nabla} P = \langle 0, 1, 1 \rangle$$

$$C(x) = x^2+y^2=5 \Rightarrow \vec{\nabla} C = \langle 2x, 2y, 0 \rangle \Rightarrow \vec{\nabla} C(1,2,1) = \langle 2, 4, 0 \rangle$$

$$\vec{\nabla} P \times \vec{\nabla} C = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = \langle -4, 2, -2 \rangle$$

$x(t) = -4t+1$

$y(t) = 2t+2$

$z(t) = -2t+1$

19) Find the points on the surface $2x^3+y-z^2=5$ at which the tangent plane is parallel to the plane $24x+y-6z=3$.

$F(x,y,z)=2x^3+y-z^2=5 \Rightarrow \vec{\nabla}F=\langle 6x^2, 1, -2z\rangle$

This is the normal vector which would define the tangent plane at the point $(x,y,z)$. We want it to be parallel to the given plane, so parallel to $\langle 24,1,-6\rangle$. We're trying to find $(x,y,z)$ to make this true.

$\langle 6x^2, 1, -2z\rangle = c\langle 24, 1, -6\rangle$ where $c$ is the parallel constant.

$$\begin{matrix} 6x^2=24c \\ 1=c \\ -2z=-6c \end{matrix} \Rightarrow \begin{matrix} 6x^2=24 \\ -2z=-6 \end{matrix} \Rightarrow \begin{matrix} x^2=4 \\ z=3 \end{matrix} \Rightarrow \begin{matrix} x=\pm 2 \\ z=3 \end{matrix}$$

Case 1: $x=2, z=3$

$2(2)^3+y-(3)^2=5 \Rightarrow y+16-9=5 \Rightarrow y=-2$ $\boxed{(2,-2,3)}$

Case 2: $x=-2, z=3$

$2(-2)^3+y-(3)^2=5 \Rightarrow y-16-9=5 \Rightarrow y=30$ $\boxed{(-2,30,3)}$

20) Let $f(x,y) = 3x^2 - 3xy^2 + y^3 + 3y^2$. Find and classify the critical points of $f$.

Critical points happen when $f_x = f_y = 0$.

$$f_x = 6x - 3y^2 = 0 \qquad\qquad 2x = y^2$$
$$\Longrightarrow \qquad\qquad \Longrightarrow \quad y^2 + 2y - y^3 = 0$$
$$f_y = -6xy + 3y^2 + 6y = 0 \qquad y^2 + 2y - 2xy = 0 \qquad -y(y^2 - y - 2) = 0$$
$$y(y-2)(y+1) = 0$$

$$\begin{array}{lcll} y=0 & & x=0 & (0,0) \\ y=2 & \xrightarrow{2x=y^2} & x=2 & (2,2) \\ y=-1 & & x=\tfrac{1}{2} & (\tfrac{1}{2},-1) \end{array}$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

To classify, we look at $f_{xx}, f_{yy}, f_{xy}$.

$f_{xx} = 6$

$f_{yy} = -6x + 6y + 6$

$f_{xy} = -6y$

Let $D = f_{xx}f_{yy} - (f_{xy})^2$. If $D<0$, saddle point

If $D>0$ & $f_{xx}>0$, minimum.

If $D>0$ & $f_{xx}<0$, maximum.

If $D=0$, can't say.

At $(0,0)$, $D=36$ & $f_{xx}=6$ so local min.

At $(2,2)$, $D=-108$, so saddle

At $(\tfrac{1}{2},-1)$, $D=-54$, so saddle

21) Find the local minimum and maximum values and saddle point(s) of the function $f(x,y) = (x^2+y^2)e^{-x}$.

$f_x = 2xe^{-x} - (x^2+y^2)e^{-x} = 0$ $\Rightarrow$ $2x = x^2+y^2$ $\Rightarrow$ $x^2-2x=0$ $\begin{cases} x=0 \\ x=2 \end{cases}$

$f_y = 2ye^{-x} = 0$ $\qquad 2y=0 \qquad y=0$

The points are $(0,0)$ & $(2,0)$.

$f_{xx} = (x^2-4x+y^2+2)e^{-x}$ $\quad f_{xx}(0,0)=2$ $\quad f_{xx}(2,0) = -2e^{-2}$

$f_{yy} = 2e^{-x}$ $\quad f_{yy}(0,0)=2$ $\quad f_{yy}(2,0) = 2e^{-2}$

$f_{xy} = -2ye^{-x}$ $\quad f_{xy}(0,0)=0$ $\quad f_{xy}(2,0)=0$

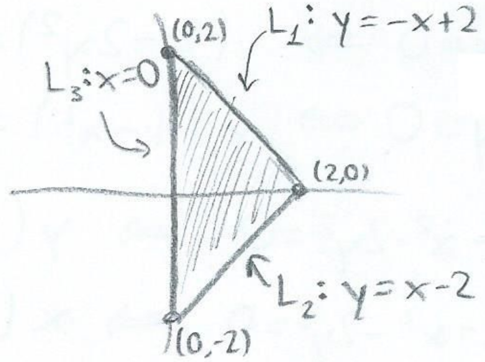
$D(0,0)=4$ & $f_{xx}(0,0)=2$ so $f(0,0)=0$ is a local min.

$D(2,0) = -4e^{-4}$ so $(2,0)$ is a saddle point.

22) Find the absolute extrema

② $f(x,y) = x^2+y^2-2x$ on the closed triangular region with vertices $(2,0)$, $(0,2)$, $(0,-2)$.

We have to check critical points and points on the boundary.




$f_x = 2x-2=0$
$f_y = 2y = 0$
$\Rightarrow$ $x=1$, $y=0$ $\boxed{(1,0) \text{ is in our region.}}$

$L_1$: $f(x,-x+2) = x^2+(2-x)^2-2x = x^2+x^2-4x+4-2x = 2x^2-6x+4$.

To find extrema, first derivative test.

$f' = 4x-6=0 \Rightarrow x = 3/2 \Rightarrow y = \frac{-3}{2}+2 = \frac{1}{2}$ $\boxed{(3/2, 1/2)}$

$L_2$: $f(x,x-2) = x^2+(x-2)^2-2x = 2x^2-6x+4$

$f' = 4x-6=0 \Rightarrow x=3/2 \Rightarrow y = 3/2-2 = -1/2$ $\boxed{(3/2,-1/2)}$

$L_3$: $f(0,y) = y^2 \Rightarrow f' = 2y=0 \Rightarrow y=0$ $\boxed{(0,0)}$

We have four points to check. You should also check corners on your region, so $(2,0)$, $(0,2)$, $(0,-2)$.

You'll see $f(0,\pm 2)=4$ maximum

$f(1,0) = -1$ minimum

② $f(x,y)=(x^2+2y^2)e^{-x^2-y^2}$ on the disk $x^2+y^2 \le 4$

$f_x = 2xe^{-x^2-y^2} - 2x(x^2+2y^2)e^{-x^2-y^2} = 0 \implies x(1-x^2-2y^2)=0$

$f_y = 4ye^{-x^2-y^2} - 2y(x^2+2y^2)e^{-x^2-y^2} = 0 \quad y(2-x^2-2y^2)=0$

$x=0 \implies y(2-2y^2)=0 \implies y=0, \pm 1$

$y=0 \implies x(1-x^2)=0 \implies x=0, \pm 1$

$\boxed{(0,0),\ (0,\pm 1) \\ (\pm 1, 0)}$

$1-x^2-2y^2=0 \implies y(1+1-x^2-2y^2)=0 \implies y(1)=0 \implies y=0$ ✓

$2-x^2-2y^2=0 \implies x(-1+2-x^2-2y^2)=0 \implies x(-1)=0 \implies x=0$ ✓

Check the boundary by method of Lagrange. $g(x,y)=x^2+y^2=4$

$\vec{\nabla} f = \lambda \vec{\nabla} g \implies$
$f_x = \lambda \cdot 2x$
$f_y = \lambda \cdot 2y$
$x^2+y^2=4$

$\implies$
$2xe^{-4} - 2x(4+y^2)e^{-4} = \lambda \cdot 2x$
$4ye^{-4} - 2y(4+y^2)e^{-4} = \lambda \cdot 2y$
$x^2+y^2=4$

$\implies$ $x(-3-y^2)e^{-4} = \lambda x$, $y(-2-y^2)e^{-4} = \lambda y$ } Let's assume $x \neq 0$ and $y \neq 0$ so we can cancel $x$ & $y$ out

$\implies$ $(-3-y^2)e^{-4} = \lambda$, $(-2-y^2)e^{-4} = \lambda$ $\implies (-3-y^2)e^{-4} = (-2-y^2)e^{-4} \implies y^2+3 = y^2+2$
$\implies 3=2$

This makes no sense, so either $x=0$ or $y=0$.

$x=0 \implies 0^2+y^2=4 \implies y=\pm 2$ $\quad \boxed{(0,\pm 2) \\ (\pm 2, 0)}$ $\quad f(0,0)=0$ min

$y=0 \implies x^2+0^2=4 \implies x=\pm 2$ $\quad f(0,\pm 1)=\frac{2}{e}$ max

③ $f(x,y)=e^{-xy}$ on $x^2+4y^2\le 1$

$f_x=-ye^{-xy}=0$ $\Longrightarrow$ $y=0$
$f_y=-xe^{-xy}=0$ $\quad\quad\quad x=0$ $\boxed{(0,0)}$

Checking the boundary, $g(x,y)=x^2+4y^2=1$.

$\nabla f=\lambda\nabla g \Longrightarrow$ $-ye^{-xy}=\lambda 2x$ $\quad \dfrac{ye^{-xy}}{x}=-2\lambda$

$\langle f_x,f_y\rangle=\lambda\langle g_x,g_y\rangle$ $\quad -xe^{-xy}=\lambda 8y \Longrightarrow$

$x^2+4y^2=1$ $\quad \dfrac{xe^{-xy}}{4y}=-2\lambda$

$\Longrightarrow \dfrac{y}{x}e^{-xy}=\dfrac{x}{4y}e^{-xy} \Longrightarrow \dfrac{y}{x}=\dfrac{x}{4y} \Longrightarrow x^2=4y^2 \Rightarrow x=\pm 2y$

$\underline{x=2y}$

$(2y)^2+4y^2=1 \Rightarrow y^2=1/8 \Longrightarrow y=\pm 1/2\sqrt{2}$ $\boxed{(\pm 1/\sqrt{2},\ \pm 1/2\sqrt{2})}$

$\underline{x=-2y}$

$(2y)^2+4y^2=1 \Longrightarrow y=\pm 1/2\sqrt{2}$ $\boxed{(\mp 1/\sqrt{2},\ \pm 1/2\sqrt{2})}$

$f\left(\mp\dfrac{1}{\sqrt{2}},\pm\dfrac{1}{2\sqrt{2}}\right)=e^{1/4}$ max

$f\left(\pm\dfrac{1}{\sqrt{2}},\pm\dfrac{1}{2\sqrt{2}}\right)=e^{-1/4}$ min

23) Find the minimum and maximum

① $f(x,y,z) = x+y+z$ constrained by $\underbrace{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}_{g(x,y,z)} = 1$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \begin{matrix} 1 = \frac{-\lambda}{x^2} \\ 1 = \frac{-\lambda}{y^2} \\ 1 = \frac{-\lambda}{z^2} \end{matrix} \Rightarrow x^2 = y^2 = z^2 \Rightarrow \begin{matrix} x = y = z \\ x = y = -z \\ x = -y = z \\ x = -y = -z \end{matrix}$$

$x=y=z \Rightarrow \frac{1}{x}+\frac{1}{x}+\frac{1}{x} = 1 \Rightarrow x = 3$ (3,3,3)

$x=y=-z \Rightarrow \frac{1}{x}+\frac{1}{x}-\frac{1}{x} = 1 \Rightarrow x = 1$ (1,1,−1)

$x=-y=z \Rightarrow \frac{1}{x}-\frac{1}{x}+\frac{1}{x} = 1 \Rightarrow x = 1$ (1,−1,1)

$x=-y=-z \Rightarrow \frac{1}{x}-\frac{1}{x}-\frac{1}{x} = 1 \Rightarrow x = -1$ (−1,1,1)

Plug each into f. Largest value is 9. Smallest is 1.

② $f(x,y,z) = x^2+y^2+z^2$ subject to $\underbrace{x-y=1}_{g}$ & $\underbrace{y^2-z^2=1}_{h}$.

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h \Rightarrow \begin{matrix} 2x = \lambda \\ 2y = -\lambda + 2\mu y \\ 2z = -2\mu z \end{matrix}$$

$2z = -2\mu z \rightarrow z = -\mu z \rightarrow z = 0$ or $\mu = -1$

$\underline{z=0}$

$y^2 - 0^2 = 1 \rightarrow y = \pm 1$: $x - 1 = 1 \rightarrow x = 2$; $x-(-1) = 1 \rightarrow x = 0$ $\boxed{\begin{matrix}(2,1,0)\\(0,-1,0)\end{matrix}}$

$\underline{\mu = -1}$

$2y = -\lambda - 2y \rightarrow 4y = -\lambda \rightarrow y = -\lambda/4$ & $x = \lambda/2 \rightarrow \frac{\lambda}{2} + \frac{\lambda}{4} = 1 \rightarrow \lambda = 4/3$

So $x = 2/3$, $y = -1/3 \rightarrow (-1/3)^2 - z^2 = 1 \rightarrow z^2 = -8/9$ impossible

$f(2,1,0) = 5$ max

$f(0,-1,0) = 1$ min

24) Find the points on the surface $x^2-yz=1$ closest to the origin.

"Closest" means "minimum distance", so we are minimizing $f(x,y,z)=x^2+y^2+z^2$ constrained by $g(x,y,z)=x^2-yz=1$.

$$\begin{aligned} 2x &= \lambda\cdot 2x \longrightarrow x=0 \text{ or } \lambda=1 \\ 2y &= \lambda\cdot(-z) \\ 2z &= \lambda\cdot(-y) \end{aligned}$$

$\underline{x=0}$

$0^2-yz=1 \to yz=-1$

$\lambda=\frac{-2y}{z}$ & $\lambda=\frac{-2z}{y} \Rightarrow \frac{-2y}{z}=\frac{-2z}{y} \Rightarrow z^2=y^2$

$\underline{x=0 \text{ \& } y=z} \Rightarrow z^2=-1$ impossible

$\underline{x=0 \text{ \& } y=-z} \Rightarrow -z^2=-1 \Rightarrow z=\pm 1$ $\boxed{\begin{matrix}(0,1,-1)\\(0,-1,1)\end{matrix}}$

$\underline{\lambda=1}$

$\begin{aligned} 2y&=-z \\ 2z&=-y \end{aligned} \Rightarrow 4yz=yz \Rightarrow yz=0 \to$ $y=0$ & $z=-2y \to z=0$; $z=0$ & $y=-2z \to y=0$

$x^2-0\cdot 0=1 \Longrightarrow x=\pm 1$ $\boxed{\begin{matrix}(1,0,0)\\(-1,0,0)\end{matrix}}$

The points $(\pm 1,0,0)$ are closer them $(0,\pm 1,\mp 1)$

25) Find the point on the ellipse $x^2+6y^2+3xy=40$ with the largest $x$-coordinate.

Maximize $f(x,y)=x$ subject to $g(x,y)=x^2+6y^2+3xy=40$

$\vec{\nabla} f=\lambda\vec{\nabla} g$

$1=\lambda(2x+3y)$

$0=\lambda(12y+3x) \longrightarrow (4y+x)\lambda=0$

$\lambda=0 \rightarrow 1=0\cdot(2x+3y)=0$ ✗

$4y+x=0 \rightarrow x=-4y$

$x^2+6y^2+3xy=40$

$\underline{x=-4y}$

$(-4y)^2+6y^2+3(-4y)y=40 \rightarrow 16y^2+6y^2-12y^2=40$

$\rightarrow 10y^2=40$

$\rightarrow y=\pm 2$

$\rightarrow x=\mp 8$

$(8,-2)$ or $(-8,2)$

So $(8,-2)$ is the point we want.

26) True / False

① There exists a function $f$ with continuous second partial derivatives such that $f_x = x + y^2$ and $f_y = x - y^2$.

False. Continuous second partial derivatives means $f_{xy} = f_{yx}$ always. But here, $f_{xy} = 2y$ & $f_{yx} = 1$ which are not always equal.

② If $f_x(a,b)$ & $f_y(a,b)$ both exist, $f$ is differentiable at $(a,b)$.

False. You would need to know that $f_x$ and $f_y$ are continuous at $(a,b)$.

③ If $f(x,y)$ is differentiable, then the rate of change of $f$ at the point $(a,b)$ in the direction of $\vec{w}$ is $\vec{\nabla} f(a,b) \cdot \vec{w}$.

False. This is true only if $\vec{w}$ is a unit vector.

④ If $f_x(a,b)=0$ and $f_y(a,b)=0$, then $f$ must have a local max or min at $(a,b)$.

False. It could be a saddle point.

⑤ If $f(x,y)$ is differentiable and $f$ has a local minimum at $(a,b)$, then $D_{\hat{u}} f(a,b)=0$ for any unit vector $\hat{u}$.

True. At a local min., $f_x=0$ & $f_y=0$, so $\vec{\nabla} f=\vec{0}$

But $D_{\hat{u}} f(a,b)=\vec{\nabla} f(a,b)\cdot\hat{u}=\vec{0}\cdot\hat{u}=0$