Exam 2 Review

I. Sequences $\{a_n\}$

Determine whether the sequence converges or diverges. If it converges, find the limit. If you used one of the theorems, Geometric Sequence Thm, Related Function Thm, Squeeze Thm, or Absolute Convergent Thm, say it.

1. $a_n = 2^{2n+1}e^{-2n}$

2. $a_n = (-1)^n \dfrac{\sqrt{n^2+1}}{\sqrt{n^2+2n-5}}$

3. $a_n = \dfrac{\sin n}{\sqrt{n+2}}$

4. $a_n = n \tan\left(\dfrac{1}{n}\right)$

5. $a_n = \left(\dfrac{n+2}{n}\right)^{3n}$

6. $a_n = \sqrt[n]{2^{1+3n}}$

7. $a_n = \dfrac{(2n)!}{2^n}$

8. $a_n = \left\{\sqrt{3}, \sqrt{\sqrt{3}}, \sqrt{\sqrt{\sqrt{3}}}\right\}$

9. Let $\{a_n\}$ be a bounded and monotonically increasing sequence such that $a_1 = \dfrac{1}{2}$ and $a_{n+1} = \dfrac{1}{1+a_n}$ for $n \geq 2$. Find the limit of the sequence.

II. Series $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

Determine whether the series converges or diverges. If it converges, find its sum.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n-1}}{5^{n-1}}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

12. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{e^n} + \frac{2}{n(n+1)}\right)$

13. $\sum_{n=3}^{\infty}\left[\arctan\left(\frac{1}{n}\right)-2\arctan\left(\frac{1}{n-1}\right)+\arctan\left(\frac{1}{n-2}\right)\right]$

14. Let S_N be the Nth partial sum of the series $\sum_{n=1}^{\infty} a_n$.

a. Suppose $S_N = 3 - \frac{N}{2^N}$, find a_n and its sum $\sum_{n=1}^{\infty} a_n$.

b. Suppose $\sum_{n=1}^{\infty} a_n = L$ where $0 < L < \infty$. Answer the next 4 questions with True, False or Not Enough Information.

i. $\lim_{n\to\infty} a_n = 0$

ii. $\lim_{N\to\infty} S_N = 0$.

iii. The series $\sum_{N=1}^{\infty} S_N$ converges.

iv. The series $\sum_{n=1}^{\infty} (a_n + 1)$ converges.

c. Answer the next 4 questions.

i. Find $\lim_{N\to\infty} S_N =$

ii. Find $\lim_{n\to\infty} a_n =$

iii. Does the series $\sum_{n\to\infty} a_n$ converge? If so, find the limit; If not, justtify.

iv. $\lim_{n\to\infty} S_N - S_{N-1} = 0$. True or False?

15. Determine the values of k for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/3}}{\sqrt{n^k + 24n}}$ will converge.

16. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

a. $\displaystyle\sum_{n=7}^{\infty} \frac{\cos n}{e^n}$ b. $\displaystyle\sum_{n=8}^{\infty} \frac{(-1)^n}{n \ln n}$

17.

a. Find an upper bound for the error when using the first four terms of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$ as an approximation of the sum.

b. How many terms do we need to add in order to estimate the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ within 0.01?

18. Which of the following statements are true?

I. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{(\ln n)^{347}}$ converges by the Direct Comparison Test.

II. $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)}{n(\ln n)^{\frac{1}{4}}}$ converges by the Alternating Series Test.

III. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^2}$ diverges by the Integral Test.

IV. $\displaystyle\sum_{n=5}^{\infty} \frac{2}{n^{1/3} \ln n}$ converges by the Direct Comparison Test.

19. Ratio and Root Tests: Which of the following series converge? Find the limit of the test you use. Is the ratio test inconclusive for any of these? In some cases you may need to use tests other than the ratio or root test.

a. $\sum_{n=1}^{\infty}(10^{1/n}-1)^n$ b. $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 5\cdot 8\cdots(3n+2)}$ c. $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ d. $\sum_{n=1}^{\infty}\frac{n^n}{2\cdot n!}$

e. $\sum_{n=1}^{\infty}\frac{n5^{2n}}{10^{n+1}}$ f. $\sum_{n=1}^{\infty}(\frac{-2n^2-1}{n^2+n+1})^{5n}$ g. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^2}$ h. $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^3}$

i. $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}$ j. $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$

20. Use the **Integral Test** to determine whether the series is convergent or divergent. If the integral test cannot be used, explain why.

a. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ b. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

c. $\sum_{n=2}^{\infty}\frac{1}{n\ln^3 n}$ d. $\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$

21. Determine whether the series is convergent or divergent using **Direct Comparison or Limit Comparison**.

a. $\sum_{n=1}^{\infty}\frac{1+\cos n}{e^n}$ b. $\sum_{n=1}^{\infty}\frac{2n^2+3n}{\sqrt{5+n^5}}$ c. $\sum_{n=1}^{\infty}\sqrt{n}\sin\left(\frac{1}{n^2}\right)$

d. $\sum_{n=1}^{\infty}\frac{\arctan(\ln n)}{n^2}$ e. $\sum_{n=1}^{\infty}\frac{\sin^2 n}{1+n^3}$ f. $\sum_{n=1}^{\infty}\frac{1}{n^3(\ln n)}$

22. Which of the following series $\sum a_n$ can we conclude converges or diverges using just the direct comparison test (DCT)? What b_n do you compare a_n to?

If DCT can not be used, what test can we use?

(You may also use abbreviations: TFD(test for divergent), AST (alternating series test), LCT (limit comparison test), INT (integral test)).

You do not have to show work.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n^2}$

i. Circle one: (convergent, divergent)

ii. If DCT: $b_n =$ ________________

iii. If Not DCT, test: ___________.

b. $\displaystyle\sum_{n=2}^{\infty} \frac{4 - \sin n}{n}$

i. Circle one: (convergent, divergent)

ii. If DCT: $b_n =$ ________________

iii. If Not DCT, test: ___________.

c. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

i. Circle one: (convergent, divergent)

ii. If DCT: $b_n =$ ________________

iii. If Not DCT, test: ___________.

d. $\displaystyle\sum_{n=2}^{\infty} \frac{e^n}{n^2}$

i. Circle one: (convergent, divergent)

ii. If DCT: $b_n =$ ________________

iii. If Not DCT, test: ___________.

23. Determine whether the series for convergence or divergence.

a. $\sum_{n=7}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$ b. $\sum_{n=8}^{\infty} \cos\left(\frac{1}{n}\right)$ c. $\sum_{n=16}^{\infty} \frac{\cos(n\pi)}{\sqrt{\ln n}}$

24. Which of the following series are divergent by the **Test for Divergence**?

a. $\displaystyle\sum_{n=7}^{\infty} \frac{1}{4 + e^{-n}}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2 - 2n + 5}$ c. $\sum_{n=8}^{\infty} \arctan n$ d. $\sum_{n=7}^{\infty} \frac{\ln n}{n}$

Exam 2 Review

1. Determine whether the **sequence** converges or diverges. If it converges, find the limit

a. $a_n = \frac{(2n-1)!}{(2n+1)!}$ b. $a_n = \frac{\cos^2 n}{2^n}$ c. $a_n = \frac{(-1)^{n+1}\sqrt{n}}{\sqrt{n}+2}$

d. $a_n = \frac{n!}{2^n}$ e. $a_n = \arctan(\ln n)$ f. $a_n = \sqrt[n]{n}$

g. $a_n = \ln(4n) - \ln(4n-1)$ h. $a_n = \frac{n}{(\ln n)^n}$ i. $a_n = (1 + \frac{3}{n})^{2n}$

2. Find the limit L of the sequence, or say DIV. (the recursive sequences (a) and (d) are convergent) .

a. $a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$ b. $\left\{\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \ldots\right\}$

c. $\left\{\frac{1}{1\cdot 3}, \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4}, \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5}, \ldots\right\}$ d. $\{\sqrt{5}, \sqrt{5\sqrt{5}}, \sqrt{5\sqrt{5\sqrt{5}}}, \ldots\}$

3.

i. What happens to the series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty} a_n = 0$?

ii. What happens to the series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty} a_n \neq 0$?

iii. Suppose $S_N = \sum_{n=1}^{N} a_n$ and that $S_N = 5 - \frac{N}{2N!}$. What can be said about $\lim_{n\to\infty} a_n$, $\sum_{N=1}^{\infty} S_N$, $\sum_{n=1}^{\infty}(a_n + 1)$? Evaluate $\sum_{i=3}^{6} a_i$.

iv Suppose $S_N = \arctan n$, then $\lim_{n\to\infty} a_n = 0$. TRUE/FALSE?

4. Suppose you know that $\sum a_n < \infty$, $\sum b_n$ diverges. Which statement below is TRUE?

i If $a_n < c_n$, then $\sum c_n$ diverges.

ii If $c_n < b_n$, then $\sum c_n$ converges.

iii If $c_n > b_n$, then $\sum c_n$ diverges.

iv If $c_n < a_n$, then $\sum c_n$ diverges.

5. $\sum \frac{42\sqrt{n}}{5n^2 - 9n + 1}$ can be shown convergent using DCT by comparing with $\sum \frac{42}{5\sqrt{n^3}}$. TRUE or FALSE?

6. Determine the value of k for which the series $\sum_{n=1}^{\infty} \frac{n^{\frac{1}{3}}}{\sqrt{n^k+24n}}$ will converge. Write your answer in interval notation.

7. Does the series converge? If so, find the sum.

a. $\sum_{n=1}^{\infty} \frac{3^{n+3}}{5^{n-1}}$ b. $\sum_{n=0}^{\infty} \frac{2^n+5^n}{2^n 5^n}$ c. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{5^{n+2}}$

d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

8. Which of the following statements are true?

I. $\sum_{n=5}^{\infty} \frac{1}{(\ln n)^{347}}$ converges by the Direct Comparison Test.

II. $\sum_{n=5}^{\infty} \frac{(-1)}{n(\ln n)^{\frac{1}{4}}}$ converges by the Alternating Series Test.

III. $\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the Direct Comparison Test.

IV. $\sum_{n=5}^{\infty} \frac{2}{n^{\frac{1}{3}} \ln n}$ converges by the Direct Comparison Test.

9. Use Root Test: Which of the following series converge? Find the correct limit of the test.

a. $\sum_{n=1}^{\infty} \left(10^{\frac{1}{n}} - 1\right)^n$ b. $\sum_{n=1}^{\infty} \left(\frac{n-7}{9n+10}\right)^n$ c. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{(\ln n)^n}$ d. $\sum_{n=1}^{\infty} \frac{n^n}{2 \cdot n!}$

e. $\sum_{n=1}^{\infty} \frac{n 5^{2n}}{10^{n+1}}$ f. $\sum_{n=4}^{\infty} \left(1 - \frac{1}{n}\right)^{5n}$ g. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ h. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

i. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$

9'. Use Ratio Test: Which of the following series converge? Find the correct limit of the test.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

10. Alternating series test. Determine if the series converges absolutely, conditionally, or diverges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n \ln n}$ b. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{(\ln n)^n}$ c. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}(\ln n)}$

11. According to the alternating series error estimation, what is the least upper estimates to the error by using the first 4 terms to approximate the sum? ($|R_4| \leq$ __)

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$

12. Determine whether the series is convergent or divergent using **Direct Comparison or Limit Comparison**.

a. $\sum_{n=1}^{\infty}\frac{1+\cos n}{e^n}$ b. $\sum_{n=1}^{\infty}\frac{n+1}{n^3+n}$ c. $\sum_{n=1}^{\infty}\sqrt{n}\tan\left(\frac{1}{n}\right)$ d. $\sum_{n=1}^{\infty}\frac{\tan\left(\frac{1}{n^2}\right)}{\ln n}$

e. $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2}$ f. $\sum_{n=1}^{\infty}\frac{n}{\ln^5 n}$ g. $\sum_{n=1}^{\infty}\frac{\ln^2 n}{n}$

13. Does the series converge? what test(s) do you use?

a. $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$ b. $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$ c. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ d. $\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$

e. $\sum_{n=1}^{\infty}\sin^n\left(\frac{1}{\sqrt{n}}\right)$ f. $\sum_{n=1}^{\infty}\sin^3\left(\frac{1}{n}\right)$

14. Which of the following series are divergent by the **Test for Divergence**?

a. $\sum_{n=1}^{\infty} n^{1/n}$ b. $\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$ c. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ d. $\sum_{n=1}^{\infty} n \tan\left(\frac{1}{n}\right)$

15. Determine whether each series is absolutely convergent, conditionally convergent or divergent. Be clear in your argument and note what test(s) you use.

a. $$\sum_{n=5}^{\infty} \frac{(-1)^n \arctan n}{n^2}$$

b. $$\sum_{n=5}^{\infty} \frac{(-1)^{n+1}}{n^{2/3} \ln^{10} n}$$

1. Determine whether each of the following series converges absolutely (A), conditionally (C) or diverges (D).

P. $\sum_{n=5}^{\infty}(-1)^n \sin\left(\frac{1}{n^5}\right)$

Q. $\sum_{n=7}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$

R. $\sum_{n=8}^{\infty}\frac{3+(-1)^n}{n}$

A. P(D), Q(D), R(D) B. P(A), Q(C), R(C) C. P(C), Q(C), R(C)

D. P(A), Q(C), R(D) E. P(D), Q(C), R(D)

2. Which of the series below can be shown divergent using TFD?

A. $\sum_{n=1}^{\infty} e^{\frac{1}{n}}$ B. $\sum_{n=1}^{\infty}\frac{1}{n}$ C. $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ D. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

3. Determine which one of the following statements is **TRUE**

I. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent

II. The series $\sum \frac{1}{n^{\sin 1}}$ is convergent

III. If a finite number of terms are added to a convergent series, then the new series still converges.

IV. If $0 < a_n < b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

A. I B. II C. III D. IV

4. Determine the **smallest** number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{7n^2}$ with an error less than 0.01.

A. 1 B. 2 C. 3 D. 4 E. 5

5. Which of the following series converge(s)?

P. $\displaystyle\sum_{n=6}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{\pi}{\sqrt{n}}\right)$

Q. $\displaystyle\sum_{n=6}^{\infty} n^3 \sin\left(\frac{1}{n}\right)$

R. $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n} \tan\left(\frac{1}{\sqrt{n}}\right)$

A. P only. B. Q only. C. R only.

D. All of them. E. None of them.

6. Let $\{a_n\}$ be a monotonically increasing sequence defined as

$$\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots \text{ and } a_n < 100.$$

Find the limit of the sequence, if it exists.

A. $\dfrac{1-\sqrt{29}}{2}$ B. 7 C. 100

D. $\dfrac{1+\sqrt{29}}{2}$ E. $\sqrt{7}$

7. Which of the following series diverges?

A. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^6 \ln n}$ B. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^{\cos(6)}}$ C. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^6}$ D. $\displaystyle\sum_{n=5}^{\infty} \frac{(-2)^n}{3^n}$

8. Which of the series below can be shown divergent using the Test for Divergence?

A. $\sum_{n=1}^{\infty} \frac{(\ln n)^4}{n}$

B. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

C. $\sum_{n=1}^{\infty} \sqrt[n]{3}$

D. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

E. $\sum_{n=1}^{\infty} \frac{1}{3n+5}$

9. Apply the Root Test to determine if each of the series below converges(C), diverges(D) or if the test is inconclusive(I).

P. $\sum_{n=8}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2}$

Q. $\sum_{n=3}^{\infty} \left(\frac{2n^2+1}{3n^2+8}\right)^{n}$

R. $\sum_{n=4}^{\infty} \frac{n^3+4}{n^{2017}+3n^2+1}$

A. P(C), Q(C), R(I)

B. P(D), Q(C), R(I)

C. P(C), Q(I), R(D)

D. P(I), Q(C), R(D)

E. P(C), Q(I), R(C)

10. Apply the **Ratio** or the **Root** test to determine if each of the following series converges(**C**), diverges(**D**) or if the test is inconclusive(**I**).

1. $\sum_{n=5}^{\infty} 8^n \left(\frac{n}{n+1}\right)^{n^2}$

2. $\sum_{n=7}^{\infty} \frac{1}{n \ln n}$

3. $\sum_{n=8}^{\infty} \left(\frac{n^2-12n+67}{2n^2+1}\right)^{n}$

A. 1D, 2D, 3C

B. 1I, 2D, 3C

C. 1D, 2I, 3D

D. 1D, 2I, 3C

E. 1C, 2I, 3D

11. Find the smallest value of N that approximates the value $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n}$ to within an error of at most 0.1. (i.e. Find N such that $|R_N| \leq 0.1$).

A. 1

B. 2

C. 3

D. 4

E. 5

12. Determine the values of p for which the series $\sum \frac{1}{n(\ln n)^p}$ is convergent.

A. $p < 0$

B. $p < 1$

C. $p > 1$

D. $p \geq 1$

E. can't be determined

13. Find the sum of the telescoping series. (**Note:** Starting index $n = 3$)

$$\sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right) \right)$$

A. -1

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. $-\frac{1}{2}$

E. $-\frac{1}{3}$

14. Determine the convergence or divergence of the series

$$\sum_{n=4}^{\infty} \frac{\sqrt{n^2+1}}{n^5 - n - 1}.$$

A. converges by Ratio test

B. converges by LCT

C. diverges by DCT

D. diverges by LCT

E. Diverges by Ratio test

1. (8 points) Given a series $\sum_{n=1}^{\infty} a_n$ and its Nth-partial sum $S_N = \sum_{n=1}^{N} a_n = \frac{2N}{9N+3}$, answer the following questions. (If a limit does not exist, or can't be determined, say so).

(a) $\lim_{N\to\infty} S_N =$ ________

(b) The series $\sum_{n=1}^{\infty} a_n$ (converges, diverges). (circle one)

Based on your answer in (b), answer exaclty one of the questions below:

If you answered that it converges, what is the sum? $\sum_{n=1}^{\infty} a_n =$ ________.

If you answered it diverges, by what test? ________

(c) $\lim_{n\to\infty} a_n =$ ________

(d) The series $\sum_{N=1}^{\infty} S_N$ (converges, diverges). (circle one)

Based on your answer in (d), answer exactly one of the questions below:

If you answered that it converges, what is the sum? $\sum_{n=1}^{\infty} S_N =$ ________.

If you answered it diverges, by what test? ____________________

(e) The series $\sum_{n=1}^{\infty}(a_n + 2)$ (converges, diverges). (circle one)

Based on your answer in (e), answer exactly one of the questions below:

If you answered it converges, what is the sum? $\sum_{n=1}^{\infty}(a_n + 2) =$ ________.

If you answered it diverges, by what test? ______________

2. (8 points) Determine if the series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n n!}{9 \cdot 16 \cdot 23 \cdot \ldots \cdot (7n+2)}$ converges or diverges.

(a) $a_n =$ ____________________

(b) $a_{n+1} =$ ____________________

(c) $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| =$

$L =$ ____________

(d) Circle one:

$L \quad (<, \quad >, \quad =) \quad 1$

(e) Which series convergence/divergence test is used here? ____________________

(f) Circle one:

The series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n n!}{9 \cdot 16 \cdot 23 \cdot \ldots \cdot (7n+2)}$ (converges, diverges, inconclusive)

3. Determine if the series converges (make sure to state the test used and check the conditions for the test)

$$\sum_{n=3}^{\infty} \frac{(2n)!}{n^{2n}}$$

1. Determine whether or not each of the following sequences converge. If the sequence does converge, find its limit. Otherwise, explain why the sequence fails to converge.

   (a) $\left\{\frac{1}{n^\pi}\right\}_{n\geq 0}$

   (b) $\left\{\sin^n\left(\frac{\pi}{7}\right)\right\}_{n\geq 0}$

   (c) $\left\{\left(\frac{1}{4}+\frac{3}{4}\right)^n\right\}_{n\geq 1}$

   (d) $\left\{\tan^{-1}\left(\frac{3n^2+1}{n!}\right)\right\}_{n\geq 1}$

   (e) $\left\{\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{(2n)!}\right\}_{n\geq 2}$

   (f) $a_1=\sqrt{2}$, $a_2=\sqrt{2\sqrt{2}}$, $a_3=\sqrt{2\sqrt{2\sqrt{2}}},\ldots$

2. Let $S_N=\sum_{n=1}^{N} a_n$. Express $S_{105}-S_{100}$ in closed form (no ellipses or $\sum$s).

3. Explore the convergence of the series

   (a) $\sum_{n=1}^{\infty}\left(\frac{1}{4}+\frac{3}{4}\right)^n$

   (b) $\sum_{n=1}^{\infty}\left[\left(\frac{1}{4}\right)^n+\left(\frac{3}{4}\right)^n\right]$

4. Use the fact that $\frac{1}{3}=0.33333333\ldots$ to express $\frac{2}{3}$ as a geometric series with $r=\frac{1}{10}$.

5. Give two examples of infinite series which pass the test for divergence, but still fail to converge.

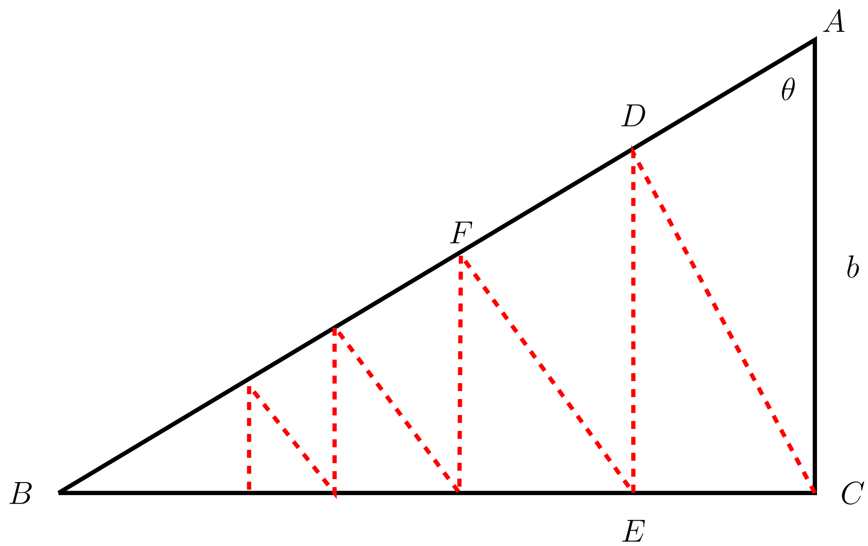
6. Consider the series $S=\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$, and it's sequence of partial sums, $S_N=\sum_{n=1}^{N}\frac{n}{(n+1)!}$.

   (a) Calculate S_1, S_2, S_3, and S_4.

   (b) Can you guess what S_5 is?

   (c) Given that $S_N=\frac{N!-1}{N!}$, what is the sum of the series S?

7. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF is perpendicular to AB, and the process is continued indefinitely (see the figure). Find the sum of the lengths of the perpendiculars.



That is, assuming this pattern continues forever, find the sum of the lengths of the dashed lines (in terms of b and θ).

8. Investigate the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{n+1}\right)$. If the series converges, find its sum.

9. Find the values of c for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{c}{n} - \frac{1}{n+1}$ converges.

10. Determine whether the series $\displaystyle\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots$ converges.

11. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 - 1}$ converges.

12. Find all positive values of b such that the series $\displaystyle\sum_{n=1}^{\infty} b^{\ln(n)}$ converges.

13. Investigate the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-2n} \right)^{2n}$

14. For what values of p does the series $\sum_{n=1}^{\infty} n^p \sin\left(\frac{1}{n^2}\right)$ converge?

15. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(n)}$ converges.

16. Find the smallest value of N so that $|S - S_N| < 0.01$ for the series $S = \sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$.

17. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1}(n)}{n^2}$ converges absolutely, converges conditionally, or diverges.

18. Suppose $\sum_{n=1}^{\infty} a_n$ converges absolutely. Is it true or false that $\sum_{n=1}^{\infty} (-1)^n a_n$ is *conditionally* convergent?

19. Investigate the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{(1.5)^n n!}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

Part I: Covers Sequence through Series Comparison Tests

1. Give an example of each of the following:

   (a) A geometric sequence:

   (b) An alternating sequence:

   (c) A sequence that is bounded, but not convergent:

   (d) A sequence that is monotonic, but not convergent:

   (e) A sequence that is not bounded and nor monotonic:

2. For each of the sequences below, determine if they are convergent. If they are, find their limit. Remember a few things:

i) If the sequence looks like something we could have taken the limit of in Calculus 1, then you can find the limit the same way you would have in Calculus 1.

ii) Factorials grow faster than polynomials and exponential functions, but not as fast as things that look like n^n.

iii) If the sequences is recursive, you usually determine convergence by trying to show the sequence is bounded and monotonic. If it is, you can then find its limit L from the recursion formula, by replacing each instance of the sequence with the limit L and then solving for L.

(a) $a_n = \dfrac{4n^3 + 2n + 4}{5n^3 - 100n + 2}$

(b) $b_n = \dfrac{2^n}{n^3}$

(c) $d_n = \ln(3n + 1) - \ln(n)$

3. What is $\sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}$? (it is bounded above)

4. For each of the series below, compute the first 5 partial sums S_1 through S_5. Do you think these series converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1,000,000}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{10^n}$

5. Which of the following would change whether or not a given series converges?

- Starting the series at $n = 5$ instead of $n = 1$.
- Deleting a million terms from the series.
- Adding a million terms to the series.

6. Does the sum of two convergent series always converge? Does the sum of two divergent series always diverge? What happens if you add a convergent series and a divergent series together?

7. For each of the following series, determine if they converge or diverge. If they converge, use the geometric series formula and/or the telescoping sum techniques to compute the sum exactly.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1+2^{n+1}}{3^n}$

   (b) $\displaystyle\sum_{n=10}^{\infty} \frac{1+4^{n-1}}{3^n}$

   (c) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\arctan n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5n+4}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{5^n} + \frac{3}{n^2+n}$

8. One application of summing geometric series is to convert repeating decimals into their fractional counterparts. For instance, the decimal $0.\overline{7}$ is really just the infinite series

$$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \dots$$

Which we can rewrite as

$$\sum_{n=1}^{\infty} \frac{7}{10^n}$$

Summing this series, we obtain that the above repeating decimal is just $\frac{7}{9}$.
Use this to find the fractional representations of the repeating decimals below:

(a) $0.\overline{9}$

(b) $0.\overline{16}$

9. The Integral Test: The integral test says that if $f(n)$ is a sequence that satisfies a couple of properties, then $\sum_{n=0}^{\infty} f(n)$ converges. What are these properties?

If $g(x)$ is continuous and decreasing:

10. If $0 \leq g(x) \leq f(x)$ and $\int_k^{\infty} g(x)\, dx$ converges, then $\sum_{n=k}^{\infty} f(n)$ converges. (T/F)

11. If $0 \leq f(x) \leq g(x)$ and $\int_k^{\infty} g(x)\, dx$ converges, then $\sum_{n=k}^{\infty} f(n)$ converges. (T/F)

12. Since $\int_1^{\infty} \frac{1}{x^2}\, dx = 1$, does this mean that $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1$? Why or why not? If not, can we find the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the techniques we learned in this class?

13. Use the Integral Test to determine whether each of these series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e}{1+n^2}$

(b) Use the result from above, determine if $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{1+n^2}$ converges.

14. The p-test: Sums of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, or $\sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^p}$ commonly appear in applications. These can be handled by the Integral Test; however, whether or not they converge due the Integral Test depends on the value of p. Rather than do the Integral Test every single time, we just remember the values of p for which these kinds of series converge. Fill in the blanks below:

- If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ____________________
- If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ____________________
- If ________, then $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ diverges.
- If ________, then $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

15. Use the p-test to quickly determine if the following integrals converge or diverge:

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

(b) $\sum_{n=8}^{\infty} \frac{1}{n^{1/2} \ln n}$

(c) $\sum_{n=8}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

(d) $\sum_{n=8}^{\infty} \frac{1}{n^2(\ln n)}$

(e) $\sum_{n=8}^{\infty} \frac{1}{n(\ln n)}$

16. **Working With Inequalities: A Primer**
When using the Direct Comparison Test to show that a series $\sum a_n$, where $a_n > 0$, converges or diverges, you are looking to find another sequence b_n, where $b_n > 0$, such that:

- $a_n \leq b_n$ and $\sum b_n$ converges. This shows that $\sum a_n$ converges.

  or that:

- $b_n \leq a_n$ and $\sum b_n$ diverges. This shows that $\sum a_n$ diverges.

But how do you show that $a_n \leq b_n$ or that $b_n \leq a_n$? Usually, you will have a guess for what b_n should be. Then, starting with a_n, you will manipulate the formula for a_n in ways that will make it either larger or smaller until you reach your target. The following page will explain some ways to do this.

How to make something bigger

- Add something positive to it (Ex: $a_n \leq a_n + 1$)
- Remove something negative from it. (Ex: $a_n - 1 \leq a_n$)
- Replace a term with something bigger. (Ex: $n^2 + n \leq n^2 + n^2 = 2n^2$)
- If you have a fraction, make the numerator larger.
- If you have a fraction, make the denominator smaller.
- If you have some other increasing function, such as a square root, you can make the stuff inside of it larger. (Ex: $\sqrt{n} < \sqrt{n+1}$).

How to make something smaller

- Add something negative to it (Ex: $a_n \geq a_n - 1$)
- Remove something positive from it. (Ex: $a_n + 1 \geq a_n$)
- Replace a term with something smaller. (Ex: $n^2 + \sin(n) \geq n^2 - 1$)
- If you have a fraction, make the numerator smaller.
- If you have a fraction, make the denominator larger.
- If you have some other increasing function, such as a square root, you can make the stuff inside of it smaller. (Ex: $\sqrt{n+1} > \sqrt{n}$).

Strategy for Direct Comparison

- Identify the terms that grow the fastest in both the numerator and denominator. Their ratio will usually form the sequence b_n. Determine if $\sum b_n$ converges or diverges.
- If b_n converges, then use the "make something bigger" strategies above to remove terms or replace terms of a_n until you get b_n.
- If b_n diverges, then use the "make something smaller" strategies above to remove terms or replace terms of a_n until you get b_n.

Example: Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{3n^3-n^2+n+1}$.

The largest term in the numerator is $\sqrt{n^2}$, and the largest in the denominator is n^3. So this should behave like $\dfrac{\sqrt{n^2}}{n^3} = \dfrac{n}{n^3} = \dfrac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges. So we want to make $a_n = \dfrac{\sqrt{n^2+1}}{3n^3-n^2+n+1}$ larger until we get to $b_n = \dfrac{1}{n^2}$.

Using our rules above, we have that

$$\begin{aligned}
a_n &= \frac{\sqrt{n^2+1}}{3n^3-n^2+n+1} \\
\text{(making numerator larger)} &\leq \frac{\sqrt{n^2+n^2}}{3n^3-n^2+n+1} \\
\text{(making denominator smaller)} &\leq \frac{\sqrt{2n^2}}{3n^3-n^2} \\
\text{(making denomiator even smaller)} &\leq \frac{\sqrt{2n^2}}{3n^3-n^3} \\
\text{(simplifying)} &= \frac{\sqrt{2}n}{2n^3} \\
&= \frac{\sqrt{2}}{n^2} \\
&= \sqrt{2}b_n
\end{aligned}$$

The $\sqrt{2}$ in front of the b_n doesn't matter, as $\sum \sqrt{2}b_n$ converges if $\sum b_n$ does. Thus, since $\sum \sqrt{2}b_n$ converges, so does $\sum a_n$.

The point of the lengthy discussion above is to show you that in order to use the direct comparison test, it is very tedious to find the correct 'size' for b_n. I hope it's enough to convenience you that, unless you NEED to use the direct comparison test when it's easy to find an upper and lower bounds for the a_n, it's much easier to use the limit comparison where you do not have to deal with the 'size'.

17. Determine the convergence of each of the following series using Direct Comparison:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + n}{2^n - n^3}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{10 - \cos n}{2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n + 4}{n \ln n}$

Limit Comparison: Sometimes, though, you will know what you want to compare your series to, but you cannot get Direct Comparison to work. (Or work easily, in any case). In this case, you should try Limit Comparison instead of Direct Comparison. The steps are pretty similar:

- Like you would for Direct Comparison, identify the terms that grow the fastest in both the numerator and denominator. Their ratio will usually form the sequence b_n. (We may phrase this as figuring out what the series "behaves like").
- Find $\lim_{n\to\infty} \frac{a_n}{b_n}$. If this limit exists, is finite, and is larger than 0, then the two series $\sum a_n$ and $\sum b_n$ both converge or both diverge. Essentially, this limit calculation shows that the two sequences decay at the same rate, and so their series will grow at the same rate.
- Limit Comparison is often better than Direct Comparison for series involving complicated rational functions, as well as series where you don't have a good guess for something to compare directly with.

18. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. Explain why this is the case.

19. Similarly, if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges. Explain why this is the case.

20. For each of the positive-term series $\sum a_n$ below, we can determine convergence by comparing to another series $\sum b_n$. Choose the sequence b_n to compare to, say whether $\sum b_n$ converges or diverges, and indicate whether you would use the Direct Comparison Test or the Limit Comparison Test to carry out the justification. Also give the inequality you will use for Direct Comparison Test, or the value of the limit, for the Limit Comparison Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 + 2n - 1}{n^5 - 2n + 3}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin\left(\frac{1}{\sqrt[3]{n}}\right)$

(f) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt[3]{n}}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^7 + 2n + 1}}$

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{\arctan n}{\sqrt{n^3 - 1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{4n^5 + n}$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5 + 3\sin n}{n^3}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{5 + 3\sin n}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \tan^3\left(\frac{1}{n}\right)$