Review2: L9-L16

1. Find the domain and the range of the given functions:
   (a) $z = f(x,y) = \sqrt{16x^2 + 4y^2 + 16}$;
   (b) $w = \ln\left(16 - x^2 - y^2 - z^2\right)$;
   (c) $z = f(x,y) = x^2 - y^2$;
   (d) $z = f(x,y) = x^2 + y^2 + 1$;
   (e) $f(x,y,z) = \sqrt{z^2 - x^2 - y^2}$.
2. Identify each of the following surfaces in $\mathbb{R}^3$ by making cross sections as a cone, ellipsoid, sphere, paraboloid, hyperbolic paraboloid, cylinder, hyperboloid of one sheet or hyperboloid of two sheets (See P. 837 of the textbook)
   (a) $x^2 + y^2 + z^2 = 4$
   (b) $2x^2 + 3y^2 + 4z^2 = 12$
   (c) $x^2 + y^2 - z = 0$
   (d) $x^2 - y^2 + z = 0$
   (e) $x^2 + y^2 = 16$
   (f) $x^2 + y^2 - z^2 = 4$
   (g) $x^2 + y^2 - z^2 = -4$
   (h) $x^2 + y^2 - z^2 = 0$
3. Describe the level surfaces of the following functions:
   (a) $f(x,y,z) = 3x^2 + 5y^2 + z^2$;
   (b) $f(x,y,z) = 2x^2 + 3y^2 - z^2$;
   (c) $f(x,y,z) = 3x^2 + 2y^2$.
4. Find the level curves of the given functions $z = f(x,y)$ for the given values of z and identify what conic sections they describe.
   (a) $z = f(x,y) = \sqrt{16x^2 + 4y^2 + 16}$, $z = 5$;
   (b) $z = f(x,y) = e^{3x^2 - 2y^2 - 6}$, $z = 3$;
   (c) $z = f(x,y) = 4x^2 + y - 3$, $z = -1$.
5. Evaluate each limit or show that it does not exist.
   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2 + y^2\right)}{x^2 + y^2}$ (b) $\lim\limits_{(x,y)\to(1,2)} \dfrac{\sqrt{y} - \sqrt{x+1}}{y - x - 1}$ (c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x - y}{\sqrt{x} - \sqrt{y}}$; $x, y > 0$
   (d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ (e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$ (f) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{xy^2 + 1} - 1}{xy^2}$

6. Find the value, if any, that can be assigned to the function

   $f(x,y)=1+e^{-\frac{1}{x^2+y^2}}$ if $(x,y)\neq(0,0)$

   at the point $(0,0)$ in order to make it continuous in $\mathbb{R}^2$.
7. Practice computing the partial derivatives of multivariable functions. Please do exercises in Section 14.3 of the textbook.
8. Use the Chain Rule to evaluate the partial derivatives of a function of several variables. Please do exercises # 1 – 12, in Section 14.5, PP 943-944 of the textbook.
9. Does there exist a function $f(x,y)$ such that $f_x=3x+y$ and $f_y=x-2y$? If yes, find f.
10. Compute the differential of each of the following functions:
    (a) $z=-x^2+2xy^2-y^3$ at the point $(1,2)$;
    (b) $z=\sqrt{x^2+y^2}$.
11. The volume of a pyramid with a square base x units on a side and a height of h is $V=\frac{1}{3}x^2h$. Approximate the change in the volume of the pyramid as the base changes from $x=2.0$ to $x=2.1$ and the height changes from $h=4.0$ to $h=3.7$.
12. Given the function $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 \text{ if } (x,y)=(0,0)\end{cases}$.
    (a) Is the function continuous at $(0,0)$?
    (b) Find $f_x(0,0)$ and $f_y(0,0)$ (if they exist) by using the definition of partial derivatives at a point.
    (c) Evaluate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ in a neighborhood of the point $(0,0)$. Are they continuous at $(0,0)$? If not, does it necessarily imply that the function is not differentiable at $(0,0)$?
    (d) Find the linear approximation $L(x,y)$ of the function at $(0,0)$.
    (e) Show that the function is differentiable at $(0,0)$ (by using the definition).
13. Given a function $f(x,y,z)=z^{1/3}\sqrt{y+\cos^2 x}$.
    (a) Find the first order partial derivatives of the function $f(x,y,z)$.
    (b) Verify whether the function is differentiable at the point $(0,0,1)$. What Theorem did you use?
    (c) Find the linearization $L(x,y,z)$ of $f(x,y,z)$ at the point $(0,0,1)$ and use it to approximate the value $f(0.01,0.02,1.03)$.
    (d) Use the differential to approximate the change of f from the point $(0,0,1)$ to $(0.01,0.02,1.03)$
14. Establish the relations between the following statements. For example, given two statements A and B. It is possible that: $A\Leftrightarrow B$ or $A\Rightarrow B$ or $B\Rightarrow A$.

A. The function $f(x,y)$ is differentiable at a point (a,b) (the definition).

B. There exist partial derivatives $f_x(a,b)$ and $f_y(a,b)$.

C. The partial derivatives are defined on an open set containing (a,b) and f_x and f_y are continuous at (a,b).

D. There exists a linearization $L(x,y)$ of the function $f(x,y)$ at (a,b).

E. There exists the tangent plane to the graph of $z = f(x,y)$ at (a,b).

F. The change of the function at the point (a,b), $\triangle z(a,b)$, can be approximated by the differential, $dz = f_x(a,b)dx + f_y(a,b)dy$.

15. Consider a hyperbolic paraboloid whose equation is $z = 4x^2 - y^2 + 2y$.
   (a) Find an equation of the tangent plane to the graph at the point $(-1,2,4)$.
   (b) Give the unit normal vector $\mathbf{N}$ to the plane at the point $(-1,2,4)$ which makes an acute angle with the positive z-axis.

16. The volume of a pyramid with a square base x units on a side and a height of h is given by $V = \frac{1}{3}x^2h$. Suppose that the dimensions are functions of time t and at a certain moment t_0 the dimensions are $x = 2\ m,\ h = 3\ m$. Let x be increasing with respect to t at a rate 0.1 m/s and h is decreasing at a rate 0.3 m/s at that instant. Find the rate at which the volume of the pyramid is changing with respect to the time at $t = t_0$. Is the volume increasing, decreasing, or neither?

17. Consider a differentiable function
   $w = f(x,y,z)$, where $x = 2uv^2,\ y = u^2 - v^2 + 2uw,\ z = uv^2w^3$.
   Find the partial derivatives $f_u,\ f_v,\ f_w$ at the point $P = (u_0, v_0, w_0)$, where $u_0 = 0,\ v_0 = 1,\ w_0 = 1$ if $f_x = a,\ f_y = b,\ f_z = c$ at the point (x_0, y_0, z_0), which corresponds to the point P.

18. Find an equation of the tangent plane, if it exists, at the point $(3,-2,2)$ on the surface defined by the equation $z^3 - xz + y = 0$ by using the property that the gradient is perpendicular to the level curve/surface. Give the normal vector.
   (Hint: assume that the given equation represents a level surface of the function $w = z^3 - xz + y$.)

19. Given a function $f(x,y) = xy - 3x^2$. Answer the questions below:
   (a) Compute the gradient vector of the function. Evaluate it at the point $(1,3)$.
   (b) Find the directional derivative $D_{\mathbf{u}}f$ at the point $(1,3)$ in the direction towards the point $(3,6)$. Does the function increase, decrease, or neither in that direction? Give a geometrical interpretation of your answer.

20. A function is defined by $x^2 + y^2 - z^2 = 1,\ z \geq 0$.
   (a) What is the graph of the function?

(b) Find a vector in the direction of the steepest ascent at the point $(1,2)$. What is the maximum rate of increase of the function at that point?

(c) Find a vector in the direction of the steepest descent at the point $(1,2)$. What is the maximum rate of decrease of the function at that point?

(d) Find a vector in a direction of no change of the function at the point $(1,2)$. Give a geometrical interpretation of your answer.

21. Test the given function on the local extrema.

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y$$

(a) Find all of the critical points.

(b) Use Sylvester Theorem to determine which of the critical points are points of local maximum, local minimum, and saddle points.

(c) Give the local maximum and minimum values, if any.

(d) Find the equations of the tangent planes at the critical points of the function $f(x,y)$. Describe these planes.

22. Find all critical points of the function $f(x,y) = \sqrt{x^2 + y^2}$, if any. Determine whether they are the points of local maximum, local minimum, or saddle points.

23. Find the absolute maximum and absolute minimum values of the function $f(x,y) = 4 - 3x - 2y$ on the closed triangular region with the vertices $(-1,0)$, $(1,2)$, and $(3,0)$.

(Hint: A linear function cannot have a max/min at an interior point of a region.)

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1. Find the maximum and minimum values, if any, of the given function subject to the given constraint.

   (a) $f(x,y) = x + 2y$ if $x^2 + y^2 = 5$;

   (b) $f(x,y) = 2x^2 - 3y^2$ if $x^2 + 2y^2 = 4$;

   (c) $f(x,y) = x^2 + y^2 - 4$ if $x^2 - y^2 = 1$;

   (d) $f(x,y,z) = x + 2y - 3z$ if $x^2 + y^2 + z^2 = 9$.

2. Find the minimum and maximum values of the function $f(x,y) = 1 - 2x^2 - y^2$ on the closed region $x^2 + y^2 \le 4$.