Review 2

1. Find the domain and range of the given functions:

(a) $z = f(x,y) = \sqrt{16x^2 + 4y^2 + 16}$

1) Domain: set $16x^2 + 4y^2 + 16 \geq 0$

The inequality holds for all $(x,y) \in \mathbb{R}^2$

$\Rightarrow$ Domain: $\boxed{\mathbb{R}^2}$

2) Range:

$f(x,y) = \sqrt{16x^2 + 4y^2 + 16} \geq \sqrt{16} = 4$

for all $(x,y) \in \mathbb{R}^2$ and $f(0,0) = 4$

$\Rightarrow$ Range: $\boxed{[4, +\infty)}$

(b) $w = \ln(16 - x^2 - y^2 - z^2)$

1) Domain: Set $16 - x^2 - y^2 - z^2 > 0$

$x^2 + y^2 + z^2 < 16$

$\Rightarrow$ Domain: $\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 < 16 \}$

2) Range: $w = f(x,y,z) = \ln(16 - x^2 - y^2 - z^2)$

$f(x,y,z) \leq \ln 16$ for all (x,y) in the Domain of f and $f(0,0,0) = \ln 16$

$\Rightarrow$ Range: $(-\infty, \ln 16]$.

(c) $z = f(x,y) = x^2 - y^2$

1) Domain: $\mathbb{R}^2$

2) Range: $\mathbb{R}$

(d) $z = f(x,y) = x^2 + y^2 + 1$

1) Domain: $\mathbb{R}^2$

2) Range:

$f(x,y) = x^2+y^2+1 \geq 1$ for all $(x,y) \in \mathbb{R}^2$

and $f(0,0) = 1$

$\Rightarrow$ Range: $[1, +\infty)$

(e) $f(x,y,z) = \sqrt{z^2-x^2-y^2}$

1) Domain: Set $z^2-x^2-y^2 \geq 0$

$$z^2 \geq x^2+y^2$$

$$|z| \geq \sqrt{x^2+y^2}$$

Domain: $\{(x,y,z) \in \mathbb{R}^3 : |z| \geq \sqrt{x^2+y^2}\}$.

2) Range: $[0, +\infty)$

2. Identify each of the following surfaces in $\mathbb{R}^3$ (by making cross sections) as a cone, ellipsoid, sphere, paraboloid, hyperbolic paraboloid, cylinder, hyperboloid of one sheet or hyperboloid of two sheets.

(a) $x^2+y^2+z^2=4$

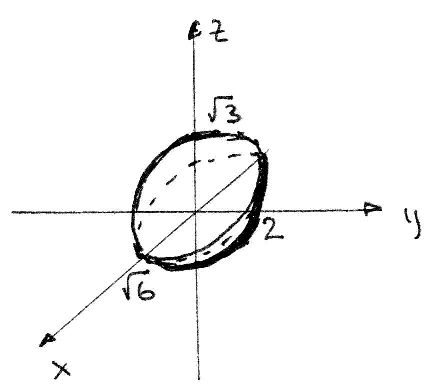
– a sphere of radius 2 centred at $(0,0,0)$.

(b) $2x^2+3y^2+4z^2=12$

$$\frac{2x^2}{12} + \frac{3y^2}{12} + \frac{4z^2}{12} = 1$$

$$\frac{x^2}{6} + \frac{y^2}{4} + \frac{z^2}{3} = 1$$

$$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{2^2} + \frac{z^2}{(\sqrt{3})^2} = 1$$



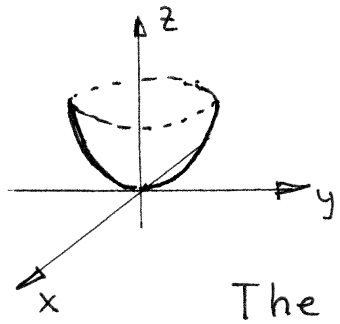
– ellipsoid

(c) $x^2 + y^2 - z = 0$

$z = x^2 + y^2$

Cross sections: $z = a \ (a > 0) \Rightarrow x^2 + y^2 = a$

– a circle



$y = 0 \Rightarrow z = x^2$

$x = 0 \Rightarrow z = y^2$

– parabolas

The surface is a paraboloid.

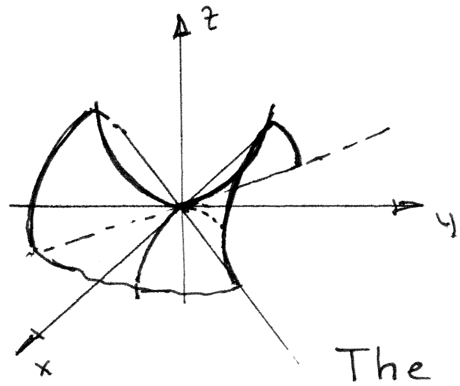
(d) $x^2 - y^2 + z = 0$

$z = y^2 - x^2$

Cross sections: $z = a \Rightarrow y^2 - x^2 = a$

– hyperbolas (for $a \neq 0$)

– lines $y = \pm x$ (for $a = 0$)



$x = 0 \Rightarrow z = y^2$

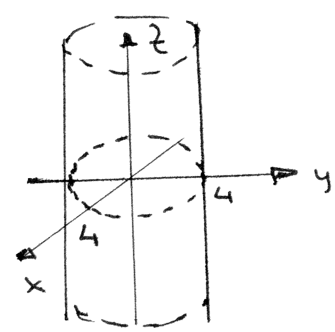
$y = 0 \Rightarrow z = -x^2$

– parabolas

The surface is a hyperbolic paraboloid.

(e) $x^2 + y^2 = 16$

– a cylinder with the axis z-axis.



Cross sections:

xz-plane: $y = 0 \Rightarrow x^2 = 16$

$x = \pm 4$

yz-plane: $x = 0 \Rightarrow y^2 = 16$

$y = \pm 4$

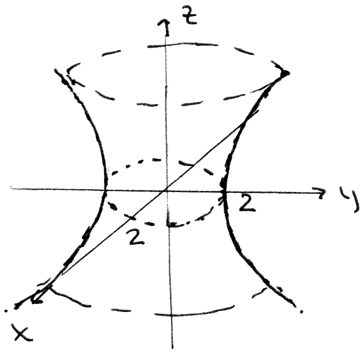
– lines

(f) $x^2 + y^2 - z^2 = 4$

Cross sections: 1) $z = 0 \Rightarrow x^2 + y^2 = 4$

$z = a \ (a \neq 0) \Rightarrow x^2 + y^2 = a^2 + 4$

– circles



2) $x = 0 \Rightarrow y^2 - z^2 = 4$

$y = 0 \Rightarrow x^2 - z^2 = 4$

– hyperbolas

The surface is a hyperboloid of one sheet.

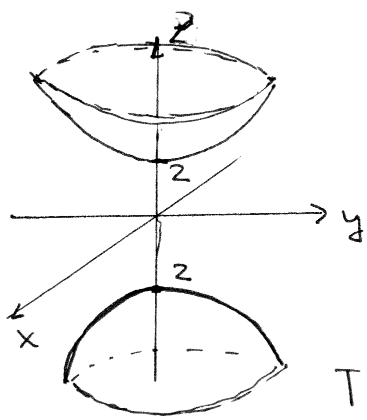
(g) $x^2 + y^2 - z^2 = -4$

$x^2 + y^2 = z^2 - 4$

$z^2 - 4 \geq 0 \iff |z| \geq 2$

Cross sections: 1) $|z| = 2 \Rightarrow (0, 0, 2), (0, 0, -2)$

– points



$|z| = a \ (a > 2) \Rightarrow x^2 + y^2 = a^2 - 4$

– circles

2) $x = 0 \Rightarrow y^2 - z^2 = -4$

$y = 0 \Rightarrow x^2 - z^2 = -4$

– hyperbolas

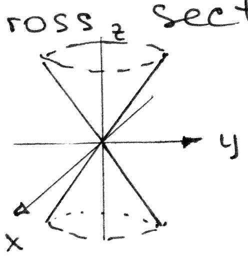
The surface is a hyperboloid of two sheets.

(h) $x^2 + y^2 - z^2 = 0$ – the surface is a cone.

$z^2 = x^2 + y^2 \iff |z| = \sqrt{x^2 + y^2} \iff z = \pm\sqrt{x^2 + y^2}$

Cross sections: $z = a \ (a \neq 0) \Rightarrow x^2 + y^2 = a^2$

– a circle



$x = 0 \Rightarrow z^2 = y^2$

$z = \pm y$

$y = 0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x$

– lines

3. Describe the level surfaces of the following functions:

(a) $f(x,y,z) = 3x^2+5y^2+z^2$

Set: $3x^2+5y^2+z^2 = a^2 \quad (a \neq 0)$

$\Rightarrow$ The level surfaces are ellipsoids.

(b) $f(x,y,z) = 2x^2+3y^2-z^2$

Set: $2x^2+3y^2-z^2 = a$

$2x^2+3y^2 = z^2+a$

The level surfaces are:

(1) elliptic hyperboloids of one sheet if $a>0$;

(2) elliptic hyperboloids of two sheets if $a<0$;

(3) a cone if $a=0$.

(c) $f(x,y,z) = 3x^2+2y^2$

Set: $3x^2+2y^2 = a^2 \quad (a \neq 0)$

$\Rightarrow$ The level surfaces are elliptic cyliders.

4. Find the level curves of the given function $z=f(x,y)$ for the given values of z and identify what conic sections they describe.

(a) $z=f(x,y)=\sqrt{16x^2+4y^2+16}\,, \quad z=5;$

Set $\sqrt{16x^2+4y^2+16} = 5$

$16x^2+4y^2+16 = 25$

$16x^2+4y^2 = 9$ (ellipse)

$$\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{y^2}{\left(\frac{3}{2}\right)^2} = 1$$

(b) $z = f(x,y) = e^{3x^2-2y^2-6}$, $z = 3$;

Set $e^{3x^2-2y^2-6} = 3$

$$3x^2-2y^2-6 = \ln 3$$

$$3x^2 - 2y^2 = 6 + \ln 3$$

– hyperbola.

(c) $z = f(x,y) = 4x^2+y-3$, $z = -1$.

Set $4x^2 + y - 3 = -1$

$$4x^2 + y = 2$$

$$y = -4x^2 + 2$$

– parabola

5. Evaluate each limit or show that it does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ (x,y)\to(0,0) \iff r\to 0 \end{array} \right|$

$$= \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} = \boxed{1}$$

(b) $\lim\limits_{(x,y)\to(1,2)} \dfrac{\sqrt{y}-\sqrt{x+1}}{y-x-1} = \lim\limits_{(x,y)\to(1,2)} \dfrac{(\sqrt{y}-\sqrt{x+1})(\sqrt{y}+\sqrt{x+1})}{(y-x-1)(\sqrt{y}+\sqrt{x+1})}$

$$= \lim_{(x,y)\to(1,2)} \frac{\cancel{y-x-1}}{(\cancel{y-x-1})(\sqrt{y}+\sqrt{x+1})} = \lim_{(x,y)\to(1,2)} \frac{1}{\sqrt{y}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{2}+\sqrt{1+1}} = \boxed{\frac{1}{2\sqrt{2}}}.$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = \lim_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y}) = \boxed{0}$$

$(x, y > 0)$

(d) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$\lim_{\substack{x\to 0\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{0}{x^2} = 0$$

$$\lim_{\substack{x\to 0\\ y=x}} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

$0 \neq \frac{1}{2} \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exists.

(e) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \left|\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ (x,y)\to(0,0) \Leftrightarrow r\to 0^+ \end{array}\right.$$

$$= \lim_{r\to 0^+} \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}} = \lim_{r\to 0^+} r\cos\theta\sin\theta = \boxed{0}$$

(f) $$\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy^2+1}-1}{xy^2} = \lim_{(x,y)\to(0,0)} \frac{(\sqrt{xy^2+1}-1)(\sqrt{xy^2+1}+1)}{xy^2(\sqrt{xy^2+1}+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{xy^2+\cancel{1}-\cancel{1}}{xy^2(\sqrt{xy^2+1}+1)} = \lim_{(x,y)\to(0,0)} \frac{\cancel{xy^2}}{\cancel{xy^2}(\sqrt{xy^2+1}+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{xy^2+1}+1} = \boxed{\frac{1}{2}}$$

6. Find the value, if any, that can be assigned to the function

$$f(x,y) = 1 + e^{-\frac{1}{x^2+y^2}} \quad \text{if} \quad (x,y) \neq (0,0)$$

at a point $(0,0)$ to make it continuous on $\mathbb{R}^2$.

1) $f(x,y)$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ by properties of continuous functions.

2) $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(1 + e^{-\frac{1}{x^2+y^2}}\right)$$

$$= 1 + \lim_{(x,y)\to(0,0)} e^{-\frac{1}{x^2+y^2}} = \left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right\}$$

$$= 1 + \lim_{r\to0^+} e^{-\frac{1}{r^2}} = 1 + 0 = 1$$

Let $f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y) = \boxed{1}$

1), 2) $\Rightarrow$ f is continuous on $\mathbb{R}^2$.

9. Does there exist a function $f(x,y)$ such that $f_x = 3x+y$ and $f_y = x-2y$?

Yes, since both functions have continuous partial derivatives on $\mathbb{R}^2$ and

$$\frac{\partial}{\partial y}(3x+y) = 1 = \frac{\partial}{\partial x}(x-2y).$$

$$f(x,y) = \int (3x+y)\,dx = \frac{3}{2}x^2 + xy + g(y)$$

$$f_y = x + g'(y) = x - 2y$$

$$g'(y) = -2y \Rightarrow g(y) = -y^2 + C, \quad C \in \mathbb{R}$$

$$\Rightarrow f(x,y) = \frac{3}{2}x^2 + xy - y^2 + C, \quad C \in \mathbb{R}$$

10. Compute the differential of each of the following functions:

(a) $z = -x^2 + 2xy^2 - y^3$ at the point $(1,2)$

$$z = f(x,y) = -x^2 + 2xy^2 - y^3$$

$$dz = f_x(1,2)\,dx + f_y(1,2)\,dy$$

$$f_x(1,2) = (-2x + 2y^2)\Big|_{(1,2)} = -2 + 8 = 6$$

$$f_y(1,2) = (4xy - 3y^2)\Big|_{(1,2)} = 8 - 12 = -4$$

$$\Rightarrow dz = 6\,dx - 4\,dy.$$

(b) $z = f(x,y) = \sqrt{x^2+y^2}$

$$dz = f_x\,dx + f_y\,dy$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$dz = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}.$$

11. The volume of the pyramid with a square base x units on a side and a height of h is $V = \frac{1}{3}x^2 h$. Approximate the change of the volume of the pyramid as the base changes from $x = 2.0$ to $x = 2.1$ and the height changes from $h = 4.0$ to $h = 3.7$

$$V = \frac{1}{3}x^2 h$$

$$\Delta V \approx dV = V_x\,dx + V_h\,dh$$

$$dx = \Delta x = 0.1\,; \quad dh = \Delta h = -0.3$$

$$V_x(2,4) = \frac{2}{3}xh \Big|_{(2,4)} = \frac{2}{3}\cdot 2\cdot 4 = \frac{16}{3}$$

$$V_h(2,4) = \frac{1}{3}x^2 \Big|_{(2,4)} = \frac{1}{3}(2)^2 = \frac{4}{3}$$

$$\Delta V \approx V_x(2,4)\,\Delta x + V_h(2,4)\,\Delta h$$
$$= \frac{16}{3}\cdot 0.1 + \frac{4}{3}(-0.3)$$
$$= \frac{4}{3}(0.4 - 0.3) = \frac{4}{3}\cdot 0.1 = \frac{4}{3}\cdot\frac{1}{10} = \boxed{\frac{2}{15}}$$

12. Given the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Is the function continuous at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{1}{x^2+y^2} =$$

$$= \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ (x,y)\to(0,0) \Leftrightarrow r\to 0^+ \end{array} \right| = \lim_{r\to 0^+} r^2\cdot\sin\frac{1}{r^2} =$$

$$= 0 = f(0,0)$$

$\Rightarrow$ f is continuous at $(0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$ (if they exist) by using the definition of partial derivatives at a point.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} =$$

$$= \lim_{h\to 0} \frac{h^2\sin\frac{1}{h^2} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h^2} = 0$$

Similarly,

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = 0$$

(c) Evaluate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ in a neighborhood of the point $(0,0)$. Are they continuous at $(0,0)$? If not, does it necessary imply that the function is not differentiable at $(0,0)$?

$$f_x = 2x \sin \frac{1}{x^2+y^2} + (x^2+y^2) \cos \frac{1}{x^2+y^2} \cdot \left(-\frac{2x}{(x^2+y^2)^2}\right)$$

$$= 2x \sin \frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2} \cos \frac{1}{x^2+y^2}.$$

$$f_y = 2y \sin \frac{1}{x^2+y^2} - \frac{2y}{x^2+y^2} \cos \frac{1}{x^2+y^2}.$$

Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} 2x \sin \frac{1}{x^2+y^2} = 0 \qquad (1)$$

$$\lim_{(x,y)\to(0,0)} 2y \sin \frac{1}{x^2+y^2} = 0. \qquad (2)$$

However, both

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+y^2} \cos \frac{1}{x^2+y^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{2y}{x^2+y^2} \cos \frac{1}{x^2+y^2}$$

do not exist. Indeed,

$$\lim_{\substack{x\to 0\\ y=0}} \frac{2x}{x^2+y^2} \cos \frac{1}{x^2+y^2} = \lim_{x\to 0} \frac{2}{x} \cos \frac{1}{x^2} \qquad (3)$$

– does not exist

$$\lim_{\substack{y\to 0\\ x=0}} \frac{2y}{x^2+y^2} \cos \frac{1}{x^2+y^2} = \lim_{y\to 0} \frac{2}{y} \cos \frac{1}{y^2} \qquad (4)$$

– does not exist.

$\Rightarrow$ From (1), (2) and (3), (4) $\Rightarrow$ both $\lim_{(x,y)\to(0,0)} f_x(x,y)$ and $\lim_{(x,y)\to(0,0)} f_y(x,y)$ do not exist.

(d) Find the linear approximation $L(x,y)$ of the function at $(0,0)$.

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$L(x,y) = 0 + 0 \cdot x + 0 \cdot y = 0$$

$$\boxed{L(x,y) = 0}$$

(e) Show that the function is differentiable at $(0,0)$ (by using the definition)

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-0)^2 + (y-0)^2}} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\frac{1}{x^2+y^2} - 0}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\sin\frac{1}{x^2+y^2} = 0$$

$\Rightarrow f(x,y)$ is differentiable at $(0,0)$.

13. Given a function $f(x,y,z) = z^{1/3}\sqrt{y + \cos^2 x}$.

(a) Find the first order partial derivatives of the function $f(x,y,z)$.

$$f_x = z^{1/3} \frac{1}{2\sqrt{y+\cos^2 x}} (2\cos x(-\sin x)) = -\frac{\cos x \sin x}{\sqrt{y+\cos^2 x}} z^{1/3}.$$

$$f_y = z^{1/3} \frac{1}{2\sqrt{y+\cos^2 x}} = \frac{z^{1/3}}{2\sqrt{y+\cos^2 x}}.$$

$$f_z = \frac{1}{3} z^{-2/3} \sqrt{y+\cos^2 x} = \frac{\sqrt{y+\cos^2 x}}{3z^{2/3}}.$$

(b) Verify whether the function is differentiable at the point $(0,0,1)$. What theorem did you use?

The partial derivatives f_x, f_y, f_z are continuous at $(0,0,1)$. Thus, the function is differentiable at that point.

(c) Find the linearization $L(x,y,z)$ of $f(x,y,z)$ at the point $(0,0,1)$ and use it to approximate the value $f(.01,.02,1.03)$

$$f_x(0,0,1) = 0$$
$$f_y(0,0,1) = \frac{1}{2}$$
$$f_z(0,0,1) = \frac{1}{3}$$
$$f(0,0,1) = 1$$

$$L(x,y,z) = f(0,0,1) + f_x(0,0,1)(x-0) + f_y(0,0,1)(y-0) + f_z(0,0,1)(z-1)$$

$$\boxed{L(x,y,z) = 1 + \frac{1}{2}y + \frac{1}{3}(z-1)}$$

$$f(.01,.02,1.03) \approx L(.01,.02,1.03)$$
$$= 1 + \frac{1}{2}(.02) + \frac{1}{3}(1.03-1) = \boxed{1.02}$$

(d) Use the differential to approximate the change of f from the point $(0,0,1)$ to $(.01,.02,1.03)$.

$$\Delta f = f(.01,.02,1.03) - f(0,0,1) \approx df(0,0,1)$$
$$df(0,0,1) = f_x(0,0,1)\,dx + f_y(0,0,1)\,dy + f_z(0,0,1)\,dz$$
$f_x = 0$, $f_y = \frac{1}{2}$, $f_z = \frac{1}{3}$, $dx = \Delta x = .01$, $dy = \Delta y = .02$, $dz = \Delta z = .03$
$$\Rightarrow \Delta f \approx 0 \cdot (.01) + \frac{1}{2}(.02) + \frac{1}{3}(.03) = \boxed{0.02}$$

15. Consider a hyperbolic paraboloid whose equation is $z = 4x^2 - y^2 + 2y$.

(a) Find an equation of the tangent plane to the graph at the point $(-1, 2, 4)$.

Let $z = f(x, y) = 4x^2 - y^2 + 2y$.

$f(-1, 2) = 4$

$f_x(-1, 2) = 8x \Big|_{(-1,2)} = -8$

$f_y(-1, 2) = (-2y + 2) \Big|_{(-1,2)} = -2.$

An equation of the tangent plane at $(-1, 2, 4)$:

$z = f(-1,2) + f_x(-1,2)(x+1) + f_y(-1,2)(y-2)$

$z = 4 - 8(x+1) - 2(y-2)$

$z = -8x - 2y$ or $\boxed{8x + 2y + z = 0}$

(b) Give the unit normal vector N to the plane at the point $(-1, 2, 4)$ which makes an acute angle with the positive z-axis.

$8x + 2y + z = 0$

$\Rightarrow$ $n = \langle 8, 2, 1 \rangle$ is a normal vector to the plane at $(-1, 2, 4)$. A unit normal vector:

$N = \dfrac{n}{\|n\|} = \dfrac{1}{\sqrt{69}} \langle 8, 2, 1 \rangle = \left\langle \dfrac{8}{\sqrt{69}}, \dfrac{2}{\sqrt{69}}, \dfrac{1}{\sqrt{69}} \right\rangle$

On the other hand,

$$N = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

where α, β, γ are the angles between N and vectors i, j, k, respectively. Since $\cos\gamma = \dfrac{1}{\sqrt{69}} > 0 \Rightarrow \gamma$ is acute.

$$\boxed{N = \frac{1}{\sqrt{69}} \langle 8, 2, 1 \rangle}.$$

In general, if $z=f(x,y)$ is differentiable at a point (x_0,y_0), a normal vector to the tangent plane to the graph of $z=f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is

$$n = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle .$$

This normal vector makes an acute angle with the positive z-axis since its third component is positive.

If $z=f(x,y) = 4x^2 - y^2 + 2y$ and $(x_0,y_0)=(-1,2)$

$$\Rightarrow n = \langle -f_x(-1,2), -f_y(-1,2), 1\rangle = \langle 8, 2, 1\rangle$$

$$\Rightarrow N = \frac{\langle -f_x, -f_y, 1\rangle}{\| \langle -f_x, -f_y, 1\rangle \|} = \frac{1}{\sqrt{69}} \langle 8, 2, 1\rangle .$$

16. The volume of a pyramid with a square base x units on a side and a height of h is given by $V = \frac{1}{3}x^2 h$. Suppose that the dimensions are functions of time t and, at a certain moment t_0, the dimensions are $x = 2m$, $h = 3m$. Let x be increasing with respect to t at a rate $0.1\ m/s$ and h is decreasing at a rate $0.3\ m/s$ at that instant. Find the rate at which the volume of the pyramid is changing with respect to the time at $t=t_0$. Is the volume increasing, decreasing, or neither?

$$V = \frac{1}{3}x^2 h ,$$

$V = V(x,h)$, where $x = x(t)$, $h = t(t)$, $(x_0, h_0) = (x(t_0), y(t_0))$
$= (2,3)$.

$$\left.\frac{dx}{dt}\right|_{t=t_0} = 0.1 \text{ (m/s)} \qquad \left.\frac{dh}{dt}\right|_{t=t_0} = -0.3 \text{ (m/s)}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$V_x(2,3) = \left.\frac{2}{3}xh\right|_{(2,3)} = \frac{2}{3}\cdot 2\cdot 3 = 4$$

$$V_h(2,3) = \left.\frac{1}{3}x^2\right|_{(2,3)} = \frac{1}{3}2^2 = \frac{4}{3}$$

$$\left.\frac{dV}{dt}\right|_{t=t_0} = V_x(2,3)\left.\frac{dx}{dt}\right|_{t=t_0} + V_h(2,3)\left.\frac{dh}{dt}\right|_{t=t_0} =$$

$$= 4\,(0.1) + \frac{4}{3}(-0.3) = 0.4 - 0.4 = \boxed{0}$$

The volume does not change at that moment.

17. Consider a differentiable function $w = f(x,y,z)$, where $x = 2uv^2$, $y = u^2 - v^2 + 2uw$, $z = uv^2w^3$. Find the partial derivatives f_u, f_v, f_w at the point $P = (u_0, v_0, w_0)$, where $u_0 = 0$, $v_0 = 1$, $w_0 = 1$ if $f_x = a$, $f_y = b$, $f_z = c$ at the point (x_0, y_0, z_0) that corresponds to the point P.

Let $\langle f_x, f_y, f_z\rangle = \langle a, b, c\rangle$ and

$$\langle x_u, y_u, z_u\rangle\big|_P = \langle 2v^2, 2u+2w, v^2w^3\rangle\big|_{(0,1,1)} = \langle 2,2,1\rangle$$

$$\langle x_v, y_v, z_v\rangle\big|_P = \langle 4uv, -2v, 2uvw^3\rangle\big|_{(0,1,1)} = \langle 0,-2,0\rangle$$

$$\langle x_w, y_w, z_w\rangle\big|_P = \langle 0, 2u, 3uv^2w^2\rangle\big|_{(0,1,1)} = \langle 0,0,0\rangle$$

Then

$$f_u(P) = f_x x_u + f_y y_u + f_z z_u = \boxed{2a + 2b + c}$$

$$f_v(P) = f_x x_v + f_y y_v + f_z z_v = \boxed{-2b}$$

$$f_w(P) = f_x x_w + f_y y_w + f_z z_w = \boxed{0}$$

18. Find an equation of the tangent plane, if it exists, at the point $(3,-2,2)$ on the surface that is defined by the equation $z^3 - xz + y = 0$ by using the property that the gradient vector is perpendicular to the level curve / surface. Give the normal vector. (Hint: assume that the given equation represents a level surface of the function $w = z^3 - xz + y$.)

Let $w = f(x,y,z) = z^3 - xz + y$.

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle -z, 1, 3z^2 - x \rangle$

$\nabla f(3,-2,2) = \langle -2, 1, 9 \rangle$

$\Rightarrow \quad n = \nabla f(3,-2,2) = \boxed{\langle -2, 1, 9 \rangle}$

is a normal vector to the surface at the point $(3,-2,2)$.

An equation of the tangent plane to the surface at the point $(3,-2,2)$:

$$n \cdot (r - r_0) = 0$$

where

$n = \langle -2, 1, 9 \rangle$, $r = \langle x, y, z \rangle$, $r_0 = \langle 3, -2, 2 \rangle$

$\langle -2, 1, 9 \rangle \cdot \langle x-3, y+2, z-2 \rangle = 0$

$-2(x-3) + y + 2 + 9(z-2) = 0$

$-2x + 6 + y + 2 + 9z - 18 = 0$

$-2x + y + 9z = 10$

$\boxed{2x - y - 9z = -10}$

19. Given a function $f(x,y) = xy - 3x^2$. Answer the questions below:

(a) Compute the gradient vector of the function. Evaluate it at the point (1,3).

$$f(x,y) = xy - 3x^2.$$

$$\nabla f = \langle f_x, f_y \rangle = \langle y - 6x, x \rangle$$

$$\nabla f(1,3) = \langle -3, 1 \rangle.$$

(b) Find the directional derivative $D_u f$ at the point (1,3) in the direction towards the point (3,6). Does the function increase, decrease, or neither in that direction. Give a geometrical interpretation of your answer.

Let $A = (1,3)$ and $B = (3,6) \Rightarrow AB = \langle 2,3 \rangle$.

$\Rightarrow u = \dfrac{AB}{\|AB\|} = \dfrac{1}{\sqrt{13}}\langle 2,3 \rangle$ is the unit vector that defines the direction from A to B.

$$D_u f(1,3) = \nabla f(1,3) \cdot u = \langle -3,1 \rangle \cdot \frac{1}{\sqrt{13}} \langle 2,3 \rangle$$

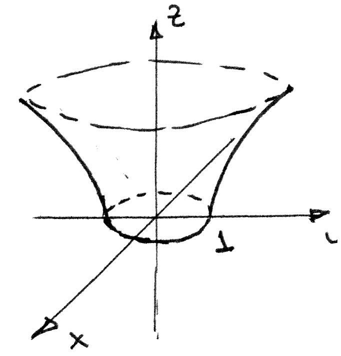
$$= \frac{1}{\sqrt{13}}(-6+3) = \boxed{-\frac{3}{\sqrt{13}}}$$

The directional derivative of f at (1,3) is the slope of the tangent line to the line of intersection of the graph of $z = f(x,y)$ with a vertical plane passing through the point (1,3,0) and parallel to the vector u. Since $D_u f(1,3) < 0$ the function is decreasing at (1,3) in that direction (defined by the vector u).

20. A function is defined by $x^2+y^2-z^2=1$, $z \geq 0$.

(a) What is the graph of the function?

The graph is an upper half of a hyperboloid of one sheet.



(b) Find a vector in the direction of the steepest ascent at the point $(1,2)$. What is the maximum rate of increase of the function at that point?

$$x^2+y^2-z^2=1, \quad z \geq 0$$

$$z^2 = x^2+y^2-1, \quad z \geq 0$$

$$z = \sqrt{x^2+y^2-1}$$

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2-1}}, \frac{y}{\sqrt{x^2+y^2-1}} \right\rangle.$$

$\nabla f(1,2) = \boxed{\langle \frac{1}{2}, 1 \rangle}$ is a vector in the direction of the steepest ascent at $(1,2)$.

The maximum rate of increase at $(1,2)$:

$$\|\nabla f(1,2)\| = \sqrt{\left(\tfrac{1}{2}\right)^2+1^2} = \sqrt{\tfrac{1}{4}+1} = \boxed{\frac{\sqrt{5}}{2}}$$

(c) Find a vector in the direction of the steepest descent at the point $(1,2)$. What is the maximum rate of decrease of the function at that point?

A vector in the direction of the steepest descent at $(1,2)$:

$$-\nabla f(1,2) = \boxed{\langle -\tfrac{1}{2}, -1 \rangle}$$

The maximum rate of decrease at $(1,2)$:

$$-\|\nabla f(1,2)\| = \boxed{-\frac{\sqrt{5}}{2}}$$

(d) Find a vector in the direction of no change of the function at the point (1,2). Give a geometrical interpretation of your answer.

A vector $\langle -1, \frac{1}{2} \rangle$ is orthogonat to $\nabla f(1,2) = \langle \frac{1}{2}, 1 \rangle$. The function does not change in that direction since $D_u f(1,2) = 0$, where u is a unit vector in the direction of $\langle -1, \frac{1}{2} \rangle$.

21. Test the given function on the local extrema.

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y.$$

(a) Find all critical points.

Set $\nabla f = 0$ or ∇f does not exist.

$$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 + 3y^2 - 15,\ 6xy - 12 \rangle$$

∇f is defined for all $(x,y) \in \mathbb{R}^2$.

$\nabla f = 0$

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0 \end{cases} \iff \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases}$$

$$y = \frac{2}{x} \implies x^2 + \frac{4}{x^2} = 5$$

$$x^4 - 5x^2 + 4 = 0, \quad x \neq 0$$

$$(x^2 - 4)(x^2 - 1) = 0$$

$$\begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases} \qquad \begin{cases} x = \pm 2 \\ y = \pm 1 \end{cases}$$

Critical points:

$P_1 = (1,2)$, $P_2 = (-1,-2)$, $P_3 = (2,1)$, $P_4 = (-2,-1)$

(b) Use Sylvester Theorem to determine which of the critical points are points of local maximum, local minimum, and saddle points.

$$f_{xx} = 6x, \quad f_{xy} = f_{yx} = 6y, \quad f_{yy} = 6x.$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix}$$

@ $P_1 = (1, 2)$:

$$\begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix}; \quad 6 > 0, \quad \begin{vmatrix} 6 & 12 \\ 12 & 6 \end{vmatrix} = 36 - 144 = -108 < 0$$

$\Rightarrow$ P_1 is a saddle point.

@ $P_2 = (-1, -2)$:

$$\begin{vmatrix} -6 & -12 \\ -12 & -6 \end{vmatrix}; \quad -6 < 0, \quad \begin{vmatrix} -6 & -12 \\ -12 & -6 \end{vmatrix} = 36 - 144 = -108 < 0$$

$\Rightarrow$ P_2 is a saddle point

@ $P_3 = (2, 1)$:

$$\begin{vmatrix} 12 & 6 \\ 6 & 12 \end{vmatrix}; \quad 12 > 0, \quad \begin{vmatrix} 12 & 6 \\ 6 & 12 \end{vmatrix} = 144 - 36 = 108 > 0$$

$\Rightarrow$ P_3 is a point of local minimum

@ $P_4 = (-2, -1)$

$$\begin{vmatrix} -12 & -6 \\ -6 & -12 \end{vmatrix}; \quad -12 < 0, \quad \begin{vmatrix} -12 & -6 \\ -6 & -12 \end{vmatrix} = 144 - 36 = 108 > 0$$

$\Rightarrow$ P_4 is a point of local maximum.

(c) $f_{loc.min} = f(2,1) = \boxed{-28}$; $\quad f_{loc.max} = f(-2,-1) = \boxed{28}$

(d) Find the equations of the tangent planes at the critical points of the function $f(x,y)$. Describe these planes.

f_x and f_y are zero at the critical points.

$\Rightarrow$ Equations of the tangent planes at $(1,2)$, $(-1,-2)$, $(2,1)$, $(-2,-1)$ are, respectively,

$$z = f(1,2) \iff \boxed{z = -26}$$

$$z = f(-1,-2) \iff \boxed{z = 26}$$

$$z = f(2,1) \iff \boxed{z = -28}$$

$$z = f(-2,-1) \iff \boxed{z = 28}$$

All these planes are parallel to the xy-plane

22. Find all critical points of the function $f(x,y) = \sqrt{x^2+y^2}$, if any. Determine whether they are the points of local maximum, local minimum, or saddle points.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

if $(x,y) \neq (0,0)$.

$\nabla f(0,0)$ does not exist.

Critical points:

$$\begin{cases} \nabla f(x,y) = 0 & \text{(no solution)} \\ \nabla f \text{ does not exist} & \iff (x,y) = (0,0) \end{cases}$$

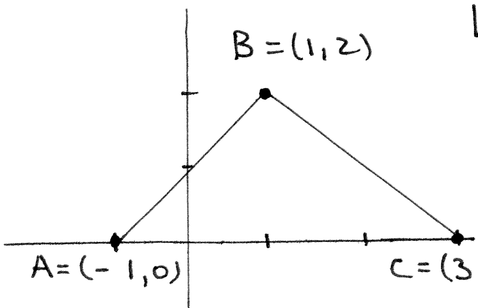
$\Rightarrow$ $\boxed{(0,0) \text{ is a critical point}}$

$f(0,0) = 0$ and $f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$.

$\Rightarrow$ $\boxed{(0,0) \text{ is a point of a local minimum}}$

(by the definition of a local minimum).

23. Find the absolute maximum and absolute minimum values of the function $f(x,y) = 4 - 3x - 2y$ on the closed triangular region with vertices $(-1,0)$, $(1,2)$, and $(3,0)$.



Let R be the triangular region bounded by the segments AB, BC, and AC. The function $f(x,y)$ is continuous on the closed and bounded region R

$\Rightarrow$ $f(x,y)$ has abs. max and abs. min on R.

1) Find all critical points of $f(x,y)$ in the interior of R, if any.

$$f(x,y) = 4 - 3x - 2y.$$

$$\begin{cases} f_x = -3 \\ f_y = -2 \end{cases} \qquad \nabla f = \langle -3, -2 \rangle \neq \langle 0, 0 \rangle$$

for all $(x,y) \in R$.

$\Rightarrow$ no critical points in R.

2) Find all critical points on the boundaries of R, if any.

(a) AB: $y = x + 1$, $x \in [-1, 1]$

$f(x, x+1) = 4 - 3x - 2(x+1) = 2 - 5x = F_1(x)$

$F_1'(x) = -5 \neq 0$ $\Rightarrow$ no critical points in AB.

(b) BC: $y = -x + 3$, $x \in [1, 3]$

$f(x, -x+3) = 4 - 3x - 2(-x+3) = -x - 2 = F_2(x)$

$F_2'(x) = -1 \neq 0$ $\Rightarrow$ no critical points in BC.

(C) AC: $y = 0$, $x \in [-1, 3]$.

$f(x, 0) = 4 - 3x = F_3(x)$

$F_3'(x) = -3 \neq 0 \Rightarrow$ no critical points.

Evaluate $f(x,y)$ at the vertices of the triangle:

$f(x,y) = 4 - 3x - 2y$.

$f(A) = f(-1, 0) = 7$

$f(B) = f(1, 2) = -3$

$f(C) = f(3, 0) = -5$

The absolute maximum of $f(x,y)$ on R is $f(A) = f(-1, 0) = \boxed{7}$

The absolute minimum of $f(x,y)$ on R is $f(C) = f(3, 0) = \boxed{-5}$.

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1. Find the maximum and minimum values, if any, of the given function subject to the given constraint.

(a) $f(x,y) = x + 2y$ if $x^2+y^2=5$.

Let $g(x,y) = x^2+y^2-5$.

Set $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$

$\nabla f = \langle 1, 2 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} 1 = \lambda(2x) & \Rightarrow \quad x = \frac{1}{2\lambda} \\ 2 = \lambda(2y) & \Rightarrow \quad y = \frac{1}{\lambda} \\ x^2+y^2 = 5 & \Rightarrow \quad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 5 \end{cases}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 5$$

$$\frac{5}{4\lambda^2} = 5 \quad \Rightarrow \quad 4\lambda^2 = 1$$

$$\lambda^2 = \frac{1}{4}$$

$$\lambda = \pm\frac{1}{2}$$

$$\Rightarrow \quad x = \frac{1}{2\lambda} = \frac{1}{2\left(\pm\frac{1}{2}\right)} = \pm 1$$

$$y = \frac{1}{\lambda} = \frac{1}{\pm\frac{1}{2}} = \pm 2$$

Critical Points: $(1, 2)$ and $(-1, -2)$.

$f(1,2) = \boxed{5}$ (the maximum value)

$f(-1,-2) = \boxed{-5}$ (the minimum value)

(b) $f(x,y) = 2x^2 - 3y^2$ if $x^2 + 2y^2 = 4$.

Let $g(x,y) = x^2 + 2y^2 - 4$.

$\nabla f = \langle 4x, -6y \rangle$

$\nabla g = \langle 2x, 4y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 4x = \lambda 2x \\ -6y = \lambda 4y \\ x^2 + 2y^2 = 4 \end{cases} \begin{array}{c} \Leftrightarrow \\ \Leftrightarrow \\ \end{array} \begin{cases} 2x(2-\lambda) = 0 \\ 2y(2\lambda+3) = 0 \\ x^2 + 2y^2 = 4 \end{cases}$$

$x = 0 \Rightarrow \left.\begin{array}{l} 2y^2 = 4 \\ y^2 = 2 \\ y = \pm\sqrt{2} \end{array}\right\} \Rightarrow$ Critical points: $(0, \pm\sqrt{2})$

$y = 0 \Rightarrow \left.\begin{array}{l} x^2 = 4 \\ x = \pm 2 \end{array}\right\} \Rightarrow$ Critical points: $(\pm 2, 0)$

$f(0, \pm\sqrt{2}) = \boxed{-6}$ (the minimum value)

$f(\pm 2, 0) = \boxed{8}$ (the maximum value).

(c) $f(x,y) = x^2 + y^2 - 4$ if $x^2 - y^2 = 1$

Let $g(x,y) = x^2 - y^2 - 1$.

$\nabla f = \langle 2x, 2y \rangle$

$\nabla g = \langle 2x, -2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2x = \lambda 2x \\ 2y = -\lambda 2y \\ x^2 - y^2 = 1 \end{cases} \iff \begin{cases} x(1-\lambda) = 0 \\ y(1+\lambda) = 0 \\ x^2 - y^2 = 1 \end{cases}$$

$x = 0 \Rightarrow -y^2 = 1 \iff y^2 = -1$ (no real solutions)

$y = 0 \Rightarrow x^2 = 1 \iff x = \pm 1$

Critical points : $(\pm 1, 0)$

$$f(\pm 1, 0) = -3.$$

$x^2 - y^2 = 1 \Rightarrow y^2 = x^2 - 1 \Rightarrow y = \pm\sqrt{x^2-1}$

$f(x, \pm\sqrt{x^2-1}) = x^2 + (x^2-1) - 4 = 2x^2 - 5$, $|x| \geq 1$

The minimum value of $(2x^2 - 5)$ for $|x| \geq 1$ is

$$f(\pm 1, 0) = \boxed{-3}$$

The maximum value of $f(x,y)$ subject to the given constraint does not exist since

$$\lim_{x \to +\infty} f(x, \pm\sqrt{x^2-1}) = \lim_{x \to +\infty} (2x^2 - 5) = +\infty$$

(d) $f(x,y,z) = x + 2y - 3z$ if $x^2 + y^2 + z^2 = 9$

Let $g(x,y,z) = x^2 + y^2 + z^2 - 9$.

$\nabla f = \langle 1, 2, -3 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases}$$

$$\begin{cases} 1 = \lambda 2x & \Rightarrow \quad x = \dfrac{1}{2\lambda} \\ 2 = \lambda 2y & \Rightarrow \quad y = \dfrac{1}{\lambda} \\ -3 = \lambda 2z & \Rightarrow \quad z = -\dfrac{3}{2\lambda} \\ x^2+y^2+z^2 = 9 & \Rightarrow \quad \left(\dfrac{1}{2\lambda}\right)^2 + \left(\dfrac{1}{\lambda}\right)^2 + \left(-\dfrac{3}{2\lambda}\right)^2 = 9 \end{cases}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 9$$

$$\frac{14}{4\lambda^2} = 9$$

$$\lambda^2 = \frac{14}{4 \cdot 9} \Rightarrow \lambda = \pm \frac{\sqrt{14}}{6}$$

$$\Rightarrow x = \frac{1}{2\lambda} = \frac{1}{2\left(\pm\frac{\sqrt{14}}{6}\right)} = \pm \frac{3}{\sqrt{14}}$$

$$y = \frac{1}{\lambda} = \frac{1}{\pm\frac{\sqrt{14}}{6}} = \pm \frac{6}{\sqrt{14}}$$

$$z = -\frac{3}{2\lambda} = -\frac{3}{2\left(\pm\frac{\sqrt{14}}{6}\right)} = \mp \frac{9}{\sqrt{14}}$$

$$f\left(\frac{3}{\sqrt{14}}, \frac{6}{\sqrt{14}}, -\frac{9}{\sqrt{14}}\right) = \frac{3}{\sqrt{14}} + \frac{12}{\sqrt{14}} + \frac{27}{\sqrt{14}} = \frac{42}{\sqrt{14}} = 3\sqrt{14}$$

$$f\left(-\frac{3}{\sqrt{14}}, -\frac{6}{\sqrt{14}}, \frac{9}{\sqrt{14}}\right) = -3\sqrt{14}$$

The maximum value: $3\sqrt{14}$

The minimum value: $-3\sqrt{14}$.

2. Find the minimum and maximum values of the function $f(x,y) = 1 - 2x^2 - y^2$ on the closed region $x^2+y^2 \leq 4$.

1) Find the critical points in $x^2+y^2 < 4$.

Set $\nabla f = 0$

$\nabla f = \langle -4x, -2y \rangle$

$$\begin{cases} -4x = 0 \\ -2y = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow (0,0) \text{ is a critical point.}$$

$$f(0,0) = \boxed{1}$$

2) Find the critical points on the boundary $x^2+y^2 = 4$.

$g(x,y) = x^2+y^2-4$.

$\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} -4x = \lambda 2x \\ -2y = \lambda 2y \\ x^2+y^2 = 4 \end{cases} \iff \begin{cases} 2x(\lambda+2) = 0 \\ 2y(\lambda+1) = 0 \\ x^2+y^2 = 4 \end{cases}$$

$x = 0 \Rightarrow y = \pm 2$

$y = 0 \Rightarrow x = \pm 2$

Critical points: $(\pm 2, 0)$, $(0, \pm 2)$

$f(0, \pm 2) = 1 - 4 = \boxed{-3}$

$f(\pm 2, 0) = 1 - 8 = \boxed{-7}$

The maximum value: $f(0,0) = 1$

The minimum value: $f(\pm 2, 0) = -7$