Review 1

1. Write an equation of the line for each of the following sets of conditions:
   (a) passing through the point $(-1,3,2)$ in the direction of the vector $\mathbf{u} = \langle 2,0,4 \rangle$;
   (b) passing through the point $(5,1,-2)$ parallel to the line $\mathbf{r}(t) = \langle 1+2t, -t, 2+4t \rangle$;
   (c) passing through the points $(-1,4,5)$ and $(2,-1,5)$;
   (d) passing through the point $(1,2,-1)$ and perpendicular to the plane $x+2y-3z=6$.
2. Write an equation of the line segment connecting the points $(-1,4,5)$ and $(2,-1,5)$.
3. Given a constant vector force $\mathbf{F} = \langle 1,2,3 \rangle$ (in Newtons).
   (a) Find the work done by the force in moving an object 2 m in the direction of the vector $\mathrm{v} = \langle 2,2,1 \rangle$.
   (b) Find the vector component of the force that does work (the component parallel to the direction of motion, $\mathbf{F}_{\parallel}$).
   (c) Find the scalar projection of the force $\mathbf{F}$ onto the direction of the motion.
   (d) Find the vector component of $\mathbf{F}$ that is perpendicular to the direction of motion, $\mathbf{F}_{\perp}$.
   (e) Find the angle θ between the force and the vector $\mathbf{v}$ of the direction of the motion.
4. Given equations of two planes in $\mathbb{R}^3$: $2x-2y+z=5$ and $x-2y=0$. Answer the questions below:
   (a) Are these planes parallel, perpendicular, coincident, or neither?
   (b) Find the angle θ between them if they are not parallel or coincident.
   (c) If they intersect, find a parametric equation of the line of intersection.
   (d) Write an equation of the plane passing through the origin and perpendicular to the line of intersection found in part (c).
5. Find an equation of the plane:
   (a) containing the points $(1,-2,4), (2,1,-3), (5,6,-2)$;
   (b) parallel to the plane $3x-2y+4z=3$ and containing the point $(-2,1,4)$;
   (c) parallel to the coordinate xz-plane and passing through the point $(-1,3-2)$.
6. Find the area S of the triangle whose vertices are $(0,1,0), (1,2,-1), (3,-2,4)$.
7. Describe the surfaces and find the indicated traces in the coordinate planes:
   (a) $x^2+y^2+z^2-2x+4y-8z=36$, $xy-$trace;
   (b) $y=x^2-2x+z^2$, $xz-$trace;
   (c) $x^2-4x+y^2+6y=24$, $xy-$trace; $yz-$trace.
8. Consider the vector function $\mathbf{r}(t) = \left\langle \frac{e^{2t}-1}{t}, \frac{\sqrt{1-t}-1}{t}, \frac{\sin(3t)}{t} \right\rangle$
   (a) Find the domain of $\mathbf{r}(t)$.
   (b) Find $\lim_{t\to 0} \mathbf{r}(t)$ if it exists.

(c) Find the set where $\mathbf{r}(t)$ is continuous.

(d) Find $\mathbf{r}'(t)$.

9. Evaluate the integral of $\mathbf{r}(t)=\left\langle e^{2t}, \sqrt{1+t}, \dfrac{t^2}{t^3+2t^2}\right\rangle$ from $t=1$ to $t=3$.

10. Find the arc length L of the portion of the curve $\mathbf{r}(t)=\langle \ln t, t^2, 2t\rangle$ that lies between the points of the intersection of the curve with the plane $y-2z+3=0$.

11. Evaluate the curvature κ for each of the given curves:

(a) $\mathbf{r}(t)=\langle \cos 2t, \sin 2t, 4t\rangle$ for any $t>0$;

(b) $\mathbf{r}(t)=\langle t^2, -3, 3t+6\rangle$ for any t.

(c) $\mathbf{r}(t)=\langle t^2-5,\ t^2+2, t\rangle$ at $t=0$.

12. Evaluate the unit vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for any t if the curve C is given by $\mathbf{r}(t)=\langle 2t+3, 6, t^2-16\rangle$.

<u>Hint</u>: Useful formulas: $\mathbf{T}=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{B}=\dfrac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}$, $\mathbf{N}=\mathbf{B}\times\mathbf{T}$.

13. Simplify: (a) $\mathbf{T}\times\mathbf{N}$; (b) $\mathbf{N}\times\mathbf{B}$; (c) $(\mathbf{T}\times\mathbf{N})\times\mathbf{B}$; (d) $(\mathbf{B}\times\mathbf{N})\times\mathbf{B}$; (e) $(\mathbf{T}\times\mathbf{N})\cdot\mathbf{B}$.

14. The trajectory of motion of an object is the curve $\mathbf{r}(t)=\langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle,\quad t\ge 0.$ Determine the following:

(a) velocity and speed at a time t;

(b) the position and velocity of the object on a long run, that is, $\lim\limits_{t\to+\infty}\mathbf{r}(t)$ and $\lim\limits_{t\to+\infty}\mathbf{v}(t)$.

(c) the length of the trajectory L from the time $t=0$ to $t=1$;

(d) the length of the trajectory $L(t)$ as a function of the time t, when $t\in[0,+\infty)$;

(e) $\lim\limits_{t\to\infty}L(t)$, that is the total length of the trajectory until the object comes to the full stop.

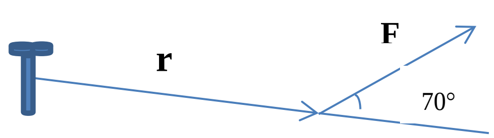
15. Find the tangential and normal components of the acceleration

(a) $\mathbf{r}(t)=\left\langle 4t, \dfrac{1}{2}(t-1)^2, t\right\rangle,\quad t\ge 0$;

(b) $\mathbf{r}(t)=\langle -\cos t, \sin t, t\rangle,\quad t\ge 0$.

<u>Hint</u>: Useful formulas: $a_T=\mathbf{a}\cdot\mathbf{T}=\dfrac{d}{dt}|\mathbf{v}|=\dfrac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$, $a_N=\mathbf{a}\cdot\mathbf{N}=\kappa|\mathbf{v}|^2=\dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$.

16. Suppose that a force of 20 N is applied to a wrench attached to a bolt in the direction perpendicular to the bolt. Find the magnitude and the direction of the torque if the force is applied at an angle of $70°$ to a wrench which is 0.15 m long.
$|\mathbf{F}|=20\ N$, $|\mathbf{r}|=0.15\ m$



Additional Problems

17. Simplify the following expressions using properties of the dot and cross products:

    (a) $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}-\mathbf{k})$; (b) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$; (c) $\mathbf{a}\times\mathbf{a}$; (d) $\mathbf{a}\cdot\mathbf{a}$; (e) $(-\mathbf{a})\times\mathbf{b}$;

    (f) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie on the same plane.

18. (a) Describe the sets of points given by the equations $x^2+y^2+z^2-2z=0$ and $x^2+y^2=1$.
    (b) Find the parametric equation of the curve of intersection of the two surfaces.
19. Find the vector projection and the scalar projection of the unit vector $\hat{\mathbf{u}}$, which is normal to the plane $3x+4z=0$ and forms an acute angle with the positive x-axis, onto a vector $\mathbf{v}$ in the xy-plane that makes the angle of $60°$ with the positive x-axis.
20. Consider the vectors:
    $\mathbf{u}=\langle -1,2,4\rangle$, $\mathbf{v}=\langle 2,-4,-8\rangle$, $\mathbf{w}=\langle 1,1,2\rangle$, and $\mathbf{p}=\langle 0,1,2\rangle$
    (a) Which pairs of the vectors are collinear (parallel), if any?
    (b) Which triples of the vectors are coplanar (lie in the same plane), if any?
    (c) Find the angles that **u** makes with the positive (a) x-axis, (b) y-axis, (c) z-axis.
21. Given a triangle whose vertices are the points $O=(0,0,0)$, $A=(2\sqrt{3},2,0)$, $B=(0,4,0)$.
    (a) Find the area S of the triangle OAB.
    (b) Given an additional point $C=(1,-2,3)$. If C is not located in the same plane as the points $O, A,$ and B, find the volume V of the parallelepiped built on the vectors $\overrightarrow{OA}, \overrightarrow{OB},$ and $\overrightarrow{OC}$.
22. Given three lines. Determine which of the lines, if any, are parallel, intersecting, or skew.

    $L_1:\ x-1=\dfrac{y}{2}=\dfrac{z-3}{2}=t \quad (-\infty<t<+\infty)$;

    $L_2:\ x=3-2s,\ y=1-4s,\ z=-1-4s \quad (-\infty<s<\infty)$;

    $L_3:\ \mathbf{r}(\tau)=\langle 3+2\tau, 4-\tau, 7+\tau\rangle \quad (-\infty<\tau<\infty)$.
23. Let L_1 be the line passing through the points $(1,-1,1)$ and $(3,-3,7)$. Let L_2 be the line passing through the points $(0,1,-1)$ and $(1,0,2)$.
    (a) Find parametric equations of the lines L_1 and L_2.
    (b) If the lines are not skew or coincident, find an equation of the plane containing both L_1 and L_2.
    (c) Find equations of two planes that are perpendicular to L_1 and are at a distance 3 from the point $(1,2,3)$.
    (d) Find the point at which the line L_1 intersects the plane $-x+3y-z=2$, if any.
24. Given three planes:
    $\mathrm{P}_1:\ 2x-y+z=1;\quad \mathrm{P}_2:\ 4x-2y+2z=1;\quad \mathrm{P}_3:\ -x+3y-z=5$
    (a) Determine which of the planes are parallel and which are intersecting.
    (b) Find the distance D_1 between the parallel planes.
    (c) Find the angle θ between the planes P_1 and P_3 if they are intersecting and an equation of the line of intersection, if any.
    (d) Find the distance D_2 from the point $(1,2,3)$ to the plane P_3.