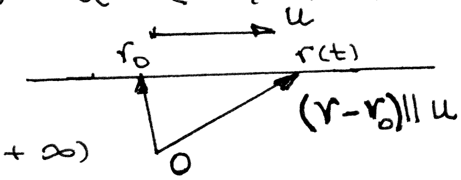
Review 1

1. Write an equation of the line for each of the following conditions:

(a) passing through the point $(-1,3,2)$ in the direction of the vector $u = \langle 2,0,4\rangle$.



$r_0 = \langle -1,3,2\rangle$, $u = \langle 2,0,4\rangle$

Equation of the line: $r - r_0 = tu$

or $r(t) = r_0 + tu$, $t \in (-\infty,+\infty)$

$\langle x(t), y(t), z(t)\rangle = \langle -1,3,2\rangle + t\langle 2,0,4\rangle$

$x(t) = -1+2t$, $y(t) = 3$, $z(t) = 2+4t$
$t \in (-\infty,+\infty)$

or

$r(t) = \langle -1+2t, 3, 2+4t\rangle$, $t \in (-\infty,+\infty)$

(b) passing through the point $(5,1,-2)$ parallel to the line $r(t) = \langle 1+2t, -t, 2+4t\rangle$

$r_0 = \langle 5,1,-2\rangle$, $u = \langle 2,-1,4\rangle$

$r(t) = \langle 5+2t, 1-t, -2+4t\rangle$, $t \in (-\infty,+\infty)$

(c) passing through the points $(-1,4,5)$ and $(2,-1,5)$

$r_0 = \langle -1,4,5\rangle$, $u = \langle 3,-5,0\rangle$

$r(t) = \langle -1+3t, 4-5t, 5\rangle$, $t \in (-\infty,+\infty)$

(d) passing through the point $(1,2,-1)$ and perpendicular to the plane $x+2y-3z=6$.

$r_0 = \langle 1,2,-1\rangle$, $u = \langle 1,2,-3\rangle$

$r(t) = \langle 1+t, 2+2t, -1-3t\rangle$, $t \in (-\infty,+\infty)$

2. Write an equation of the line segment connecting the points $(-1,4,5)$ and $(2,-1,5)$

$r(t) = r_0 + tu$, $t \in [0,1]$

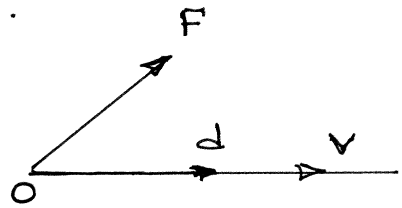
$r_0 = \langle -1,4,5\rangle$, $u = \langle 3,-5,0\rangle$

$r(t) = \langle -1+3t, 4-5t, 5\rangle$, $t \in [0,1]$

3. Given a constant vector force

$$F = \langle 1, 2, 3 \rangle \quad \text{(in Newtons)}$$

(a) Find the work done by the force in moving an object 2m in the direction of the vector $v = \langle 2, 2, 1 \rangle$.



Work : $W = F \cdot d$

where d is the displacement.

$$d = 2 \cdot \frac{v}{\|v\|} = 2 \cdot \frac{\langle 2, 2, 1 \rangle}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3} \langle 2, 2, 1 \rangle$$

$$\Rightarrow W = F \cdot d = \frac{2}{3} \langle 1, 2, 3 \rangle \cdot \langle 2, 2, 1 \rangle = \frac{2}{3}(2 + 4 + 3)$$

$$= \frac{2}{3} \cdot 9 = \boxed{6 \ N \cdot m}$$

(b) Find the vector component of the force that does work (the component parallel to the direction of the motion, $F_{\parallel}$)

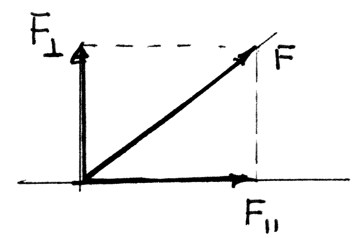
$$F_{\parallel} = \text{proj}_v F = \frac{F \cdot v}{\|v\|^2} v = \frac{\langle 1, 2, 3 \rangle \cdot \langle 2, 2, 1 \rangle}{2^2 + 2^2 + 1} v$$

$$= \frac{9}{9} v = v = \langle 2, 2, 1 \rangle$$

(c) Find the scalar projection of the force onto the direction of the motion.

$$\text{scal}_v F = \frac{F \cdot v}{\|v\|} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 2, 2, 1 \rangle}{3} = \frac{9}{3} = \boxed{3}$$

(d) Find the vector component of the force that is perpendicular to the direction of the motion, $F_\perp$.



$F_\perp = F - F_\parallel = \langle 1, 2, 3 \rangle - \langle 2, 2, 1 \rangle$

$= \langle -1, 0, 2 \rangle.$

(e) Find the angle Θ between the force and the vector $\mathbf{v}$ of the direction of the motion.

$$\cos\Theta = \frac{F \cdot v}{\|F\|\,\|v\|} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 2, 2, 1 \rangle}{\sqrt{1^2+2^2+3^2} \cdot \sqrt{2^2+2^2+1^2}} =$$

$$= \frac{2+4+3}{3\sqrt{14}} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

$$\Rightarrow \quad \Theta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

4. Given equations of two planes in R^3:

(1) $2x - 2y + z = 5$ and (2) $x - 2y = 0$

Answer the questions below:

(a) Are these planes parallel, perpendicular, coincident, or neither?

Normal vectors to the planes are:

$n_1 = \langle 2, -2, 1 \rangle$, $n_2 = \langle 1, -2, 0 \rangle$

$n_1 \cdot n_2 = 2 + 4 + 0 = 6 \neq 0$

$\Rightarrow$ the planes are not perpendicular.

$n_1 \nparallel n_2$ since none of the vectors is a scalar multiple of the other

$\Rightarrow$ the planes are neither parallel nor coincident.

Answer: $\boxed{\text{Neither}}$

(b) Find the angle Θ between them if they are not parallel or coincident.

$$\cos\Theta = \frac{|n_1 \cdot n_2|}{\|n_1\|\,\|n_2\|} = \frac{|\langle 2,-2,1\rangle \cdot \langle 1,-2,0\rangle|}{\sqrt{2^2+(-2)^2+1^2}\cdot\sqrt{1^2+(-2)^2+0^2}} =$$

$$= \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\Rightarrow \boxed{\Theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)}$$

(c) If they intersect, find a parametric equation of the line of intersection.

$$r(t) = r_0 + tu, \quad t \in (-\infty, +\infty)$$

$$u \parallel (n_1 \times n_2)$$

$$n_1 = \langle 2, -2, 1\rangle$$

$$n_2 = \langle 1, -2, 0\rangle$$

$$n_1 \times n_2 = \langle 2, 1, -2\rangle$$

Let $u = \langle 2, 1, -2\rangle$

To find r_0: set $x = 0$ in $\begin{cases} 2x - 2y + z = 5 \\ x - 2y = 0 \end{cases}$

$\Rightarrow y = 0, \quad z = 5$

and $r_0 = \langle 0, 0, 5\rangle$

$$\Rightarrow \boxed{r(t) = \langle 2t, t, 5-2t\rangle, \quad t \in (-\infty, +\infty)}$$

(d) Write an equation of the plane passing through the origin and perpendicular to the line of intersection found in part (c).

$$n \cdot (r - r_0) = 0$$

$n = \langle 2, 1, -2\rangle$, $r_0 = \langle 0, 0, 0\rangle$, $r = \langle x, y, z\rangle$

$$\langle 2, 1, -2\rangle \cdot \langle x-0, y-0, z-0\rangle = 0$$

$$\boxed{2x + y - 2z = 0}$$

5. Find an equation of the plane:

(a) containing points $(1,-2,4), (2,1,-3), (5,6,-2)$

Let $A=(1,-2,4)$, $B=(2,1,-3)$, $C=(5,6,-2)$

$AB = \langle 1, 3, -7 \rangle$, $AC = \langle 4, 8, -6 \rangle$

$n_1 = \langle 1, 3, -7 \rangle$ and $n_2 = \langle 2, 4, -3 \rangle$

$(n_1 \times n_2) \perp$ plane

$n_1 \times n_2 = \langle 19, -11, -2 \rangle$

Let $n = n_1 \times n_2 = \langle 19, -11, -2 \rangle$

$r_0 = \langle 1, -2, 4 \rangle$, $r = \langle x, y, z \rangle$

$n \cdot (r - r_0) = 0$ - an equation of the plane

$\langle 19, -11, -2 \rangle \cdot \langle x-1, y+2, z-4 \rangle = 0$

$19(x-1) - 11(y+2) - 2(z-4) = 0$

$\boxed{19x - 11y - 2z = 33}$

(b) parallel to the plane $3x - 2y + 4z = 3$ and containing the point $(-2, 1, 4)$

$n = \langle 3, -2, 4 \rangle$, $r_0 = \langle -2, 1, 4 \rangle$

$n \cdot (r - r_0) = 0$

$\langle 3, -2, 4 \rangle \cdot \langle x+2, y-1, z-4 \rangle = 0$

$3(x+2) - 2(y-1) + 4(z-4) = 0$

$\boxed{3x - 2y + 4z = 8}$

(c) parallel to coordinate xz-plane and passing through the point $(-1, 3, -2)$

$n = \langle 0, 1, 0 \rangle$, $r_0 = \langle -1, 3, -2 \rangle$

$n \cdot (r - r_0) = 0$

$\langle 0, 1, 0 \rangle \cdot \langle x+1, y-3, z+2 \rangle = 0$

$y - 3 = 0$ or $\boxed{y = 3}$

6. Find the area S of the triangle whose vertices are $(0,1,0)$, $(1,2,-1)$, $(3,-2,4)$.

Let $A=(0,1,0)$, $B=(1,2,-1)$, $C=(3,-2,4)$

$$AB = \langle 1, 1, -1 \rangle$$

$$AC = \langle 3, -3, 4 \rangle$$

$$AB \times AC = \langle 1, -7, -6 \rangle$$

$$S = \frac{1}{2} \| AB \times AC \| = \frac{1}{2} \sqrt{1^2 + (-7)^2 + (-6)^2} = \boxed{\frac{\sqrt{86}}{2}}$$

7. Describe the surfaces and find the indicated traces in the coordinate planes.

(a) $x^2+y^2+z^2-2x+4y-8z=36$

$$(x^2-2x+1)+(y^2+4y+4)+(z^2-8z+16)=36+1+4+16$$

$$(x-1)^2+(y+2)^2+(z-4)^2=57$$

– a sphere of radius $\sqrt{57}$ and center $(1,-2,4)$.

<u>xy-trace:</u> Set $z=0$

$$\Rightarrow (x-1)^2+(y+2)^2=41, \quad z=0$$

(b) $y = x^2 - 2x + z^2$

$$y = (x^2-2x+1)+z^2-1 \iff y = (x-1)^2+z^2$$

or $y+1 = (x-1)^2+z^2$

– a paraboloid

<u>xz-trace:</u> Set $y=0$

$$(x-1)^2+z^2=1, \quad y=0$$

(c) $x^2 - 4x + y^2 + 6y = 24$

$(x^2 - 4x + 4) + (y^2 + 6y + 9) = 24 + 4 + 9$

$(x-2)^2 + (y+3)^2 = 37$

- cylinder

xy - trace: $(x-2)^2 + (y+3)^2 = 37, \quad z = 0$

yz - trace: Set $x = 0$

$\Rightarrow \quad 4 + (y+3)^2 = 37$

$(y+3)^2 = 33$

$y + 3 = \pm\sqrt{33}$

$y = -3 \pm \sqrt{33}$

Lines: $x = 0, \quad y = -3 \pm \sqrt{33}, \quad z = t$
$t \in (-\infty, +\infty)$

8. Consider the function:

$$r(t) = \left\langle \frac{e^{2t} - 1}{t}, \frac{\sqrt{1-t} - 1}{t}, \frac{\sin 3t}{t} \right\rangle.$$

(a) Find the domain or $r(t)$.

$t \neq 0$ and $1 - t \geq 0 \sim t \leq 1$

Domain: $(-\infty, 0) \cup (0, 1]$

(b) Find $\lim\limits_{t \to 0} r(t)$ if it exists.

$$\lim_{t\to 0} r(t) = \left\langle \lim_{t\to 0} \frac{e^{2t}-1}{t}, \lim_{t\to 0} \frac{\sqrt{1-t}-1}{t}, \lim_{t\to 0} \frac{\sin 3t}{t} \right\rangle$$

L'H rule
$$= \left\langle \lim_{t\to 0} \frac{2e^{2t}}{1}, \lim_{t\to 0} \frac{-\frac{1}{2\sqrt{1-t}}}{1}, \lim_{t\to 0} \frac{3\cos 3t}{1} \right\rangle$$

$$= \left\langle 2, -\frac{1}{2}, 3 \right\rangle$$

(c) $r'(t) = \left\langle \left(\dfrac{e^{2t}-1}{t}\right)', \left(\dfrac{\sqrt{1-t}-1}{t}\right)', \left(\dfrac{\sin 3t}{t}\right)' \right\rangle$

$$= \left\langle \frac{2e^{2t}\cdot t - (e^{2t}-1)}{t^2}, \frac{-\frac{t}{2\sqrt{1-t}} - (\sqrt{1-t}-1)}{t^2}, \frac{3\cos 3t\cdot t - \sin 3t}{t^2} \right\rangle$$

$$= \left\langle \frac{2te^{2t} - e^{2t} + 1}{t^2}, \frac{t-2+2\sqrt{1-t}}{2t^2\sqrt{1-t}}, \frac{3t\cos 3t - \sin 3t}{t^2} \right\rangle.$$

9. Evaluate the integral of

$r(t) = \left\langle e^{2t}, \sqrt{1+t}, \dfrac{t^2}{t^3+2t^2} \right\rangle$ from $t=1$ to $t=3$.

$$\int_1^3 r(t)\,dt = \left\langle \int_1^3 e^{2t}dt, \int_1^3 \sqrt{1+t}\,dt, \int_1^3 \frac{t^2}{t^3+2t^2}dt \right\rangle$$

$$\int_1^3 e^{2t}dt = \frac{e^{2t}}{2}\Big|_1^3 = \frac{1}{2}(e^6 - e^2)$$

$$\int_1^3 \sqrt{1+t}\,dt = \frac{2}{3}(1+t)^{3/2}\Big|_1^3 = \frac{2}{3}\left(4^{3/2} - 2^{3/2}\right) = \frac{2}{3}(8 - 2\sqrt{2}$$

$$\int_1^3 \frac{t^2}{t^3+2t}dt = \int_1^3 \frac{dt}{t+2} = \ln|t+2|\Big|_1^3 = \ln 5 - \ln 3 = \ln\frac{5}{3}$$

$$\Rightarrow \int_1^3 r(t)\,dt = \left\langle \frac{1}{2}(e^6-e^2), \frac{2}{3}(8-2\sqrt{2}), \ln\frac{5}{3} \right\rangle$$

10. Find the length L of the portion of the curve $r(t) = \langle \ln t, t^2, 2t \rangle$ that lies between the points of intersection of the curve with the plane $y - 2z + 3 = 0$

$r(t) = \langle x(t), y(t), z(t) \rangle \Rightarrow x(t) = \ln t$, $y(t) = t^2$, $z(t) = 2$

Substituting $y(t)$ and $z(t)$ into $y - 2z + 3 = 0$

we have:

$$t^2 - 4t + 3 = 0$$
$$(t-3)(t-1) = 0$$
$$t = 3 \quad \text{and} \quad t = 1$$

$\Rightarrow$ The curve intersects the plane at the points that correspond to $t = 1, 3$.

$$\Rightarrow \quad L = \int_1^3 \|r'(t)\|\,dt$$

$$r'(t) = \left\langle \frac{1}{t},\ 2t,\ 2 \right\rangle$$

$$\|r'(t)\| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \sqrt{\left(\frac{1}{t} + 2t\right)^2} = \left|\frac{1}{t} + 2t\right|$$

$$1 \leq t \leq 3 \quad \Rightarrow \quad \|r'(t)\| = \frac{1}{t} + 2t$$

$$L = \int_1^3 \left(\frac{1}{t} + 2t\right) dt = \ln|t|\Big]_1^3 + t^2\Big]_1^3 =$$

$$= \ln 3 - \ln 1 + (9-1) = \boxed{\ln 3 + 8}$$

11. Evaluate the curvature k for each of the given curves.

$$k = \frac{\|T'(t)\|}{\|r'(t)\|}$$

(a) $r(t) = \langle \cos 2t, \sin 2t, 4t \rangle, \quad t > 0$

$$r'(t) = \langle -2\sin 2t,\ 2\cos 2t,\ 4 \rangle$$

$$\|r'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 16} =$$

$$= \sqrt{4(\sin^2 2t + \cos^2 2t) + 16} = \sqrt{4 + 16} =$$

$$= \sqrt{20} = 2\sqrt{5}$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{2\sqrt{5}} \langle -2\sin 2t, 2\cos 2t, 4 \rangle$$

$$= \frac{1}{\sqrt{5}} \langle -\sin 2t, \cos 2t, 2 \rangle$$

$$T'(t) = \frac{1}{\sqrt{5}} \langle -2\cos 2t, -2\sin 2t, 0 \rangle$$

$$= \frac{2}{\sqrt{5}} \langle -\cos 2t, -\sin 2t, 0 \rangle$$

$$\|T'(t)\| = \frac{2}{\sqrt{5}} \sqrt{\cos^2 2t + \sin^2 2t} = \frac{2}{\sqrt{5}}.$$

$$\Rightarrow \quad k = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{2}{\sqrt{5}} \, \frac{1}{2\sqrt{5}} = \boxed{\frac{1}{5}}$$

(b) $r(t) = \langle t^2, -3, 3t+6 \rangle$ for any t.

$$k = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$r'(t) = \langle 2t, 0, 3 \rangle$$

$$r''(t) = \langle 2, 0, 0 \rangle$$

$$r'(t) \times r''(t) = \langle 0, 6, 0 \rangle$$

$$\|r'(t) \times r''(t)\| = 6$$

$$\|r'(t)\| = \sqrt{(2t)^2 + 3^2} = (4t^2+9)^{1/2}$$

$$\boxed{k = \frac{6}{(4t^2+9)^{3/2}}}$$

(C) $r(t) = \langle t^2-5, t^2+2, t \rangle$ at $t = 0$

$$r'(t) = \langle 2t, 2t, 1 \rangle$$

$$r'(0) = \langle 0, 0, 1 \rangle$$

$$r''(t) = \langle 2, 2, 0 \rangle$$

$$r'(0) \times r''(0) = \langle -2, 2, 0 \rangle$$

$$\| r'(0) \times r''(0) \| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\| r'(0) \| = 1$$

$$k = \frac{\| r'(0) \times r''(0) \|}{\| r'(0) \|} = \frac{2\sqrt{2}}{1} = \boxed{2\sqrt{2}}$$

12. Evaluate the unit vectors T, N, B for any t if the curve C is given by

$$r(t) = \langle 2t+3,\ 6,\ t^2-16 \rangle$$

$$r'(t) = \langle 2,\ 0,\ 2t \rangle$$

$$\| r'(t) \| = \sqrt{2^2 + (2t)^2} = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$T(t) = \frac{r'(t)}{\| r'(t) \|} = \boxed{\left\langle \frac{1}{\sqrt{1+t^2}},\ 0,\ \frac{t}{\sqrt{1+t^2}} \right\rangle}$$

$$B = \frac{r'(t) \times r''(t)}{\| r'(t) \times r''(t) \|}$$

$$r'(t) = \langle 2,\ 0,\ 2t \rangle$$

$$r''(t) = \langle 0,\ 0,\ 2 \rangle$$

$$r'(t) \times r''(t) = \langle 0,\ -4,\ 0 \rangle$$

$$\| r'(t) \times r''(t) \| = 4$$

$$B(t) = \frac{1}{4} \langle 0, -4, 0 \rangle = \boxed{\langle 0, -1, 0 \rangle}$$

$$N(t) = B(t) \times T(t) = \boxed{\left\langle \frac{-t}{\sqrt{1+t^2}},\ 0,\ \frac{1}{\sqrt{1+t^2}} \right\rangle}$$

13. (a) $T \times N = B$

(b) $N \times B = T$

(c) $(T \times N) \times B = B \times B = 0$

(d) $(B \times N) \times B = -(N \times B) \times B = -T \times B = B \times T = N$

or $(B \times N) \times B = B \times (N \times B) = B \times T = N$

(e) $(T \times N) \cdot B = B \cdot B = 1$

14. The trajectory of motion of an object is the curve $r(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t} \rangle$

Determine the following:

(a) Velocity and speed at a time t:

$v(t) = r'(t) =$

$= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), e^{-t} \rangle$

Speed:

$\| v(t) \| = v(t) =$

$= e^{-t}\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2 + 1}$

$= e^{-t}\sqrt{3}.$

(b) $\lim\limits_{t \to +\infty} r(t) = \langle \lim\limits_{t \to +\infty} e^{-t}\cos t, \lim\limits_{t \to +\infty} e^{-t}\sin t, \lim\limits_{t \to +\infty} e^{-t} \rangle$

$= \langle 0, 0, 0 \rangle$

$\lim\limits_{t \to +\infty} v(t) = \langle 0, 0, 0 \rangle$

(c) The length of the trajectory L from the time $t = 0$ to $t = 1$:

$$L = \int_0^1 \| v(t) \| dt = \sqrt{3} \int_0^1 e^{-t} dt = \sqrt{3}\,(-e^{-t})\Big|_0^1 =$$

$$= \sqrt{3}\,(1 - e^{-1}) = \sqrt{3}\,(1 - \tfrac{1}{e})$$

(d) $L(t) = \int_0^t \| v(u) \| du = \sqrt{3} \int_0^t e^{-u} du = \sqrt{3}\,(-e^{-u})\Big|_0^t =$

$= \sqrt{3}\,(1 - e^{-t})$

(e) $\lim\limits_{t \to +\infty} L(t) = \lim\limits_{t \to +\infty} \sqrt{3}\,(1 - e^{-t}) = \sqrt{3}$

15. Find the tangental and normal components of the acceleration.

(a) $r(t) = \langle 4t, \frac{1}{2}(t-1)^2, t \rangle$, $t \geq 0$

$$a(t) = a_T T + a_N N$$

$$a_T = \frac{v \cdot a}{\| v \|} = \frac{r' \cdot r''}{\| r' \|} \; ; \quad a_N = \frac{\| v \times a \|}{\| v \|} = \frac{\| r' \times r'' \|}{\| r' \|}$$

$$v(t) = r'(t) = \langle 4, t-1, 1 \rangle$$

$$\| v(t) \| = \sqrt{4^2 + (t-1)^2 + 1^2} = \sqrt{17 + (t-1)^2}$$

$$a(t) = r''(t) = \langle 0, 1, 0 \rangle$$

$$a_T = \frac{v \cdot a}{\| v \|} = \boxed{\frac{t-1}{\sqrt{17 + (t-1)^2}}}$$

$$a_N = \frac{\| v \times a \|}{\| v \|} = \boxed{\frac{\sqrt{17}}{\sqrt{17 + (t-1)^2}}}$$

$$v(t) = \langle 4, t-1, 1 \rangle$$

$$a(t) = \langle 0, 1, 0 \rangle$$

$$v \times a = \langle -1, 0, 4 \rangle$$

$$\| v \times a \| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}.$$

(b) $r(t) = \langle -\cos t, \sin t, t \rangle$, $t \geq 0$

$$v(t) = r'(t) = \langle \sin t, \cos t, 1 \rangle \; ; \quad \| v(t) \| = \sqrt{2}$$

$$a(t) = r''(t) = \langle \cos t, -\sin t, 0 \rangle$$

$$v(t) \times a(t) = \langle \sin t, \cos t, -1 \rangle$$

$$\| v(t) \times a(t) \| = \sqrt{\sin^2 + \cos^2 t + 1} = \sqrt{2}$$

$$v(t) \cdot a(t) = 0$$

$$\Rightarrow a_T = \frac{v(t) \cdot a(t)}{\| v \|} = \boxed{0} \; ; \quad a_N = \frac{\| v(t) \times a(t) \|}{\| v(t) \|} = \frac{\sqrt{2}}{\sqrt{2}} = \boxed{1}$$

16. Suppose that a force of 20 N is applied to a wrench attached to a bolt in the direction perpendicular to the bolt. Find the magnitute and the direction of the torque if the force is applied at an angle of $70°$ to a wrench which is 0.15 m long.

Torque: $\tau = r \times F$

$\|\tau\| = \|r\| \, \|F\| \sin \theta$

$\|F\| = 20\,N$, $\|r\| = 0.15\ m$, $\theta = 70°$

$\|\tau\| = 0.15 \cdot 20 \cdot \sin 70° = \boxed{2.81\ N\cdot m}$

The direction of torque is upward.

17. Simplify the following expressions using properties of the dot and cross products:

(a) $(i+j)\times(j-k) = i\times j - i\times k + j\times j - j\times k$

$= k + j + 0 - i = -i + j + k$

(b) $(a\times b)\cdot a = \mathbf{0}$ since $a\times b \perp a$

(c) $a \times a = \mathbf{0}$

(d) $a \cdot a = \|a\|^2$

(e) $(-a)\times b = -(a\times b) = b \times a$

(f) $u\cdot(v\times w) = 0$ since $u \perp v\times w$

18. (a) Describe the set of points given by the equations $x^2+y^2+z^2-2z=0$ and $x^2+y^2=1$

(1) $x^2+y^2+(z-1)^2 = 1$ - a sphere of radius 1 with the center at $(0,0,1)$

(a) $x^2+y^2=1$

- a cylinder whose axis is the z-axis.

(b) $x^2+y^2=1 \quad \Rightarrow \quad x^2+y^2+z^2-2z=0$

$$1+z^2-2z=0$$
$$(z-1)^2=0$$
$$z=1$$

$\Rightarrow$ The curve of intersection is a circle of radius 1 center $(0,0,1)$ located in the plane $z=1$:

$$x=\cos t\ ,\quad y=\sin t\ ,\quad z=1 \qquad (0\le t\le 2\pi)$$

19. Find the vector projection and the scalar projection of the unit vector $\mathbf{u}$, which is normal to the plane $3x+4z=0$ and forms an acute angle with the positive x-axis, onto a vector $\mathbf{v}$ in the xy-plane that makes the angle of $60°$ with the positive x-axis.

$3x+4z=0 \quad \Rightarrow \quad n=\langle 3,0,4\rangle$ is a normal vector to the plane

Let θ be the angle between n and i.

$$\cos\theta=\frac{n\cdot i}{\|n\|\|i\|}=\frac{3}{5}>0 \Rightarrow \theta \text{ is acute}$$

$$\Rightarrow \quad u=\frac{n}{\|n\|}=\left\langle \frac{3}{5},0,\frac{4}{5}\right\rangle$$

$$V=\langle \cos 60°, \sin 60°, 0\rangle=\left\langle \frac{1}{2},\frac{\sqrt{3}}{2},0\right\rangle$$

$$\operatorname{proj}_V u=\frac{u\cdot V}{\|V\|}\frac{V}{\|V\|}=(u\cdot V)V=\frac{3}{10}V=\boxed{\left\langle \frac{3}{20},\frac{3\sqrt{3}}{20},0\right\rangle}$$

$$\operatorname{scal}_V u=\frac{u\cdot V}{\|V\|}=u\cdot V=\boxed{\frac{3}{10}}$$

20. Consider the vectors:

$u = \langle -1, 2, 4 \rangle$, $v = \langle 2, -4, -8 \rangle$, $w = \langle 1, 1, 2 \rangle$

$p = \langle 0, 1, 2 \rangle$

(a) Which pairs of vectors are collinear (parallel), if any?

$$V = -2u$$

$\Rightarrow$ $\boxed{\text{u and v are parallel}}$

(b) Which triples are coplanar (lie in the same plane), if any?

$$u \parallel v \Rightarrow u \times v = 0$$

$$\Rightarrow (u \times v) \cdot w = 0 \quad \text{and} \quad (u \times v) \cdot p = 0$$

$\Rightarrow$ $\boxed{u, v, w}$ lie in the same plane

$\boxed{u, v, p}$ lie in the same plane.

$$(u \times w) \cdot p = \langle 0, 6, -3 \rangle \cdot \langle 0, 1, 2 \rangle = 6 - 6 = 0$$

$\Rightarrow$ $\boxed{u, w, p}$ lie in the same plane

$$(v \times w) \cdot p = 0$$

$\Rightarrow$ $\boxed{v, w, p}$ lie in the same plane

(c) Find the angles that u makes with the positive (a) x-axis, (b) y-axis, (c) z-axis.

(a) $\cos \alpha = \dfrac{u \cdot \hat{i}}{\|u\| \, \|i\|} = -\dfrac{1}{\sqrt{21}} \Rightarrow \alpha = \cos^{-1}\left(-\dfrac{1}{\sqrt{21}}\right)$

$\cos \beta = \dfrac{u \cdot \hat{j}}{\|u\| \, \|j\|} = \dfrac{2}{\sqrt{21}} \Rightarrow \beta = \cos^{-1}\left(\dfrac{2}{\sqrt{21}}\right)$

$\cos \gamma = \dfrac{u \cdot k}{\|u\| \, \|k\|} = \dfrac{4}{\sqrt{21}} \Rightarrow \gamma = \cos^{-1}\left(\dfrac{4}{\sqrt{21}}\right)$

21. Given a triangle whose vertices are the points $O = (0,0,0)$, $A = (2\sqrt{3}, 2, 0)$, $B = (0,4,0)$

(a) Find the area S of the triangle OAB.

$$OA = \langle 2\sqrt{3}, 2, 0 \rangle$$

$$OB = \langle 0, 4, 0 \rangle$$

$$OA \times OB = \langle 0, 0, 8\sqrt{3} \rangle$$

$$\|OA \times OB\| = 8\sqrt{3}$$

$$S = \frac{1}{2} \| OA \times OB \| = \boxed{4\sqrt{3}}$$

(b) $C = (1, -2, 3)$. If C is not located in the same plane as O, A, and B, find the volume V of the parallelepiped built on the vectors OA, OB, and OC.

$$OC = \langle 1, -2, 3 \rangle$$

$$(OA \times OB) \cdot OC = \langle 0, 0, 8\sqrt{3} \rangle \langle 1, -2, 3 \rangle = 24\sqrt{3} \neq 0$$

$\Rightarrow$ C is not in the same plane with O, A, and B.

$$V = |(OA \times OB) \cdot OC| = \boxed{24\sqrt{3}}.$$

22. Given three lines. Determine which lines, if any, are parallel, intersecting, or skew.

$$L_1: \frac{x-1}{1} = \frac{y}{2} = \frac{z-3}{2} = t \quad (-\infty < t < +\infty)$$

$$\sim \quad x = 1 + t, \quad y = 2t, \quad z = 3 + 2t$$

$$V_1 = \langle 1, 2, 2 \rangle$$

$L_2: \ x = 3 - 2s, \ y = 1 - 4s, \ z = -1 - 4s$
$(-\infty < s < +\infty)$

$v_2 = \langle -2, -4, -4 \rangle$

$L_3: \ r(t) = \langle 3 + 2\tau, 4 - \tau, 7 + \tau \rangle$ $(-\infty < \tau < +\infty)$

$v_3 = \langle 2, -1, 1 \rangle.$

1) $v_2 = -2v_1 \implies v_1 \parallel v_2$

The line L_1 passes through the point $(1,0,3)$ that is, $x = 1$, $y = 0$, $z = 3$, but L_2 does not pass through that point since the system

$$\begin{cases} 1 = 3 - 2s \\ 0 = 1 - 4s \\ 3 = -1 - 4s \end{cases}$$

has no solution.

$\implies$ The lines $\boxed{L_1 \text{ and } L_2 \text{ are parallel}}$

2) Vector v_3 is not multiple of either v_1 or v_2 $\implies$ L_3 is not parallel to either L_1 or L_2

Check if L_3 intersects L_1. Set

$$\begin{cases} 1 + t = 3 + 2\tau & \implies t = 2\tau + 2 \\ 2t = 4 - \tau & \implies 4\tau + 4 = 4 - \tau \\ 3 + 2t = 7 + \tau & \quad 5\tau = 0 \implies \tau = 0, \ t = 2 \end{cases}$$

Substituting $\tau = 0$, $t = 2$ into the last equation:

$$3 + 4 = 7 + 0$$
$$7 = 7 \quad \checkmark$$

$\implies$ The lines L_1 and L_3 are intersecting.

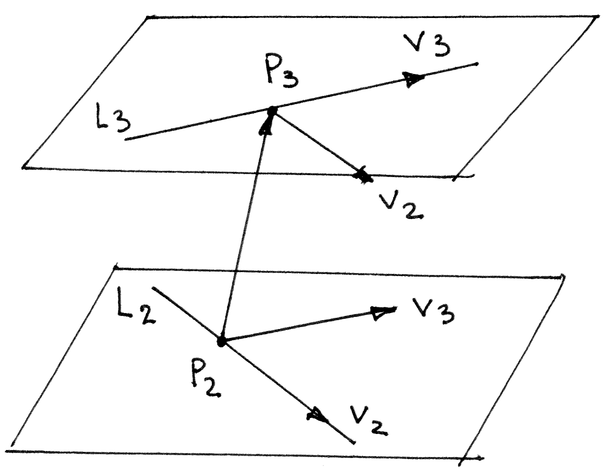
To find the point of intersection, set $\tau = 0$

in $r(\tau) = \langle 3+3\tau, 4-\tau, 7+\tau \rangle$

$r(0) = \langle 3, 4, 7 \rangle \Rightarrow$ Point of intersection: $\boxed{(3,4,7)}$

3) Show that L_2 and L_3 are skew, that is,

$L_2 \nparallel L_3$ and $L_2 \cap L_3 = \emptyset$



For L_2: $P_2 = (3, 1, -1)$

$V_2 = \langle -2, -4, -4 \rangle$

For L_3: $P_3 = (3, 4, 7)$

$V_3 = \langle 2, -1, 1 \rangle$

The lines L_2 and L_3 are skew $\Leftrightarrow$ vectors V_2, V_3, and P_2P_3 are not coplanar, or

equivalently,

$$P_2P_3 \cdot (V_2 \times V_3) \neq 0$$

$V_2 = \langle -2, -4, -4 \rangle$

$V_3 = \langle 2, -1, 1 \rangle$

$V_2 \times V_3 = \langle -8, -6, 10 \rangle = (-2) \langle 4, 3, -5 \rangle$

$P_2P_3 = \langle 0, 3, 8 \rangle$

$P_2P_3 \cdot (V_2 \times V_3) = (-2) \langle 0, 3, 8 \rangle \cdot \langle 4, 3, -5 \rangle$

$= (-2)(9 - 40) = (-2)(-31) = 62 \neq 0$

$\Rightarrow L_2$ and L_3 are skew.

23. Let L_1 be the line passing through the points $(1,-1,1)$ and $(3,-3,7)$. Let L_2 be the line passing through the points $(0,1,-1)$ and $(1,0,2)$.

(a) Find parametric equations of the lines L_1 and L_2.

For L_1: $\quad P_1 = (1,-1,1)\,,\quad Q_1 = (3,-3,7)$

$$P_1Q_1 = \langle 2,-2,6\rangle \Rightarrow v_1 = \langle 1,-1,3\rangle$$

$$r(t) = \langle 1+t,\,-1-t,\,1+3t\rangle\,,\qquad t\in(-\infty,+\infty)$$

or $\quad x = 1+t,\quad y = -1-t,\quad z = 1+3t\,,\quad t\in(-\infty,+\infty)$

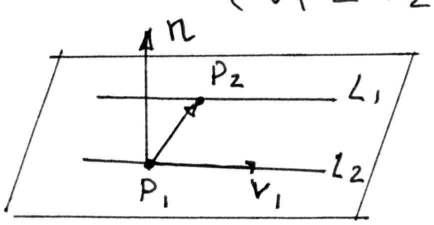
For L_2: $\quad P_2 = (0,1,-1)\,,\quad Q_2 = (1,0,2)$

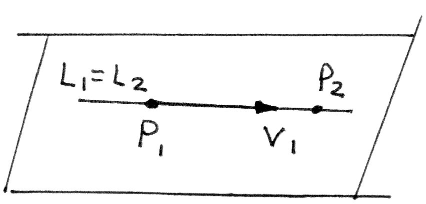
$$P_2Q_2 = \langle 1,-1,3\rangle \Rightarrow v_2 = \langle 1,-1,3\rangle$$

$$r(s) = \langle s,\,1-s,\,-1+3s\rangle\,,\qquad s\in(-\infty,+\infty)$$

or $\quad x = s,\quad y = 1-s,\quad z = -1+3s\,,\quad s\in(-\infty,+\infty)$

(b) $v_1 \parallel v_2 \Rightarrow L_1 \parallel L_2$ or $L_1 = L_2$
$(v_1 = v_2)$





$$P_1P_2 = \langle -1,2,-2\rangle$$
$$v_1 = \langle 1,-1,3\rangle$$

$$P_1P_2\times v_1 = \langle 4,1,-1\rangle \neq 0$$

$\Rightarrow L_1 \parallel L_2$

$\Rightarrow n = P_1P_2\times v_1$ is perpendicular to the plane containing L_1 and L_2.

Let $P_1 = (1,-1,1)\,,\quad n = \langle 4,1,-1\rangle$

Equation of the plane:

$$4(x-1)+(y+1)-(z-1) = 0$$

$$\boxed{4x+y-z=2}$$

(c) Find equations of two planes that are perpendicular to L_1 and are at a distance 3 from the point $(1, 2, 3)$.

Let $P = (1, 2, 3)$, $n = v_1 = \langle 1, -1, 3 \rangle$

An equation of a plane perpendicular to L_1 is:

$$1 \cdot x + (-1)y + 3z = d$$

$$x - y + 3z = d$$

The distance from the plane $x - y + 3z = d$ to the point $P = (1, 2, 3)$ is:

$$D = \frac{|d - (1 - 2 + 3(3))|}{\|n\|} = \frac{|d - 8|}{\sqrt{11}}$$

$$D = 3 \iff \frac{|d-8|}{\sqrt{11}} = 3$$

$$|d - 8| = 3\sqrt{11}$$

$$d - 8 = \pm 3\sqrt{11}$$

$$d = 8 \pm 3\sqrt{11}$$

$\Rightarrow$ The equations of two planes are:

$$x - y + 3z = 8 + 3\sqrt{11}$$

$$x - y + 3z = 8 - 3\sqrt{11}$$

(d) Find the point at which the line L_1 intersects the plane $-x + 3y - z = 2$, if any.

L_1: $r(t) = \langle 1+t, -1-t, 1+3t \rangle$, $t \in (-\infty, +\infty)$.

or $x = 1+t$, $y = -1-t$, $z = 1+3t$, $t \in (-\infty, +\infty)$

We substitute x, y, z into $-x + 3y - z = 2$

$$-(1+t) + 3(-1-t) - (1+3t) = 2$$

$$-7t = 7 \Rightarrow t = -1$$

$\Rightarrow x = 1 - 1 = 0$, $y = -1 - (-1) = 0$, $z = 1 + 3(-1) = -2$

Point: $\boxed{(0, 0, -2)}$

24. Given three planes:

$P_1: 2x - y + z = 1$; $P_2: 4x - 2y + 2z = 1$;

$P_3: -x + 3y - z = 5$.

(a) $n_1 = \langle 2, -1, 1 \rangle$, $n_2 = \langle 4, -2, 2 \rangle$

$n_2 = 2n_1$ (1)

$P_1: 2x - y + z = d_1$, $d_1 = 1$

$P_2: 4x - 2y + 2z = d_2$, $d_2 = 1$

$d_2 \neq 2d_1$ (2)

(1), (2) $\Rightarrow$ $\boxed{P_1 \parallel P_2.}$

$n_3 = \langle -1, 3, -1 \rangle$

$n_3 \nparallel n_1$ and $n_3 \nparallel n_2$

$\Rightarrow$ $\boxed{P_3 \cap P_1 \neq \varnothing}$ and $\boxed{P_3 \cap P_2 \neq \varnothing}$

(b) Find the distance D_1 between the parallel planes.

$P_2: 4x - 2y + 2z = 1$ $\sim$ $P_2: 2x - y + z = \frac{1}{2}$.

$n_1 = \langle 2, -1, 1 \rangle$

$$D_1 = \frac{|d_1 - \frac{1}{2}|}{\|n_1\|}$$

$$D_1 = \frac{|1 - \frac{1}{2}|}{\sqrt{6}} = \boxed{\frac{1}{2\sqrt{6}}}$$

(c) Find the angle θ between the planes P_1 and P_3 if they are intersecting and an equation of the line of intersection, if any.

$P_1: \; 2x - y + z = 1 \quad \Rightarrow \quad n_1 = \langle 2, -1, 1 \rangle$

$P_3: \; -x + 3y - z = 5 \quad \Rightarrow \quad n_3 = \langle -1, 3, -1 \rangle$

$$\cos\theta = \frac{|n_1 \cdot n_3|}{\|n_1\| \, \|n_3\|} = \frac{|-2-3-1|}{\sqrt{6}\,\sqrt{11}} = \frac{6}{\sqrt{6}\,\sqrt{11}} = \frac{\sqrt{6}}{\sqrt{11}} = \sqrt{\frac{6}{11}}$$

$$\boxed{\theta = \cos^{-1}\left(\sqrt{\frac{6}{11}}\right)}$$

To find an equation of the line of intersection of P_1 and P_3, we need a direction vector v and a point P.

A direction vector v is perpendicular to both n_1 and n_3, thus, v is parallel to $n_1 \times n_3$.

$$n_1 = \langle 2, -1, 1 \rangle$$
$$n_3 = \langle -1, 3, -1 \rangle$$
$$n_1 \times n_3 = \langle -2, 1, 5 \rangle \quad \Rightarrow \quad V = \langle -2, 1, 5 \rangle$$

To find a point P on the line, we look for a solution of the system

$$\begin{cases} 2x - y + z = 1 \\ -x + 3y - z = 5 \end{cases}$$

Set $y = 0 \quad \Rightarrow \quad \begin{cases} 2x + z = 1 \\ -x - z = 5 \end{cases}$

$$x = 6, \quad z = -11$$

$$\Rightarrow \quad P = (6, 0, -11)$$

An equation of the line:

$$r(t) = \langle 6 - 2t, t, -11 + 5t \rangle, \quad t \in (-\infty, +\infty)$$

(d) Find the distance D_2 from the point $(1, 2, 3)$ to the plane P_3.

$P_3: \ -x + 3y - z = 5 \quad \Rightarrow \quad n_3 = \langle -1, 3, -1 \rangle$

$$D_2 = \frac{|5 - (-1 + 3(2) - 3)|}{\|n_3\|} = \boxed{\frac{3}{\sqrt{11}}}$$